Third Way Consistency of 3D YM and Gravity

Paul K. Townsend

University of Cambridge

Based principally on

- Minimal Massive 3D Gravity, Bergshoeff, Hohm, Merbis,Routh, PKT, arXiv:1404.2867
- Yang-Mills as massive Chern-Simons theory, Arvanitakis, Sevrin, PKT, arXiv:1501.07548
- Exotic Massive 3D Gravity, Ozkan, Pang, PKT, arXiv:1806.04179

POTOR5, September 26, 2018

Consistency for 3D YM

$SU(2)$ Yang-Mills : $\quad F = dA + A \times A\,, \qquad \tilde{F} := \star F$

$$L = \frac{1}{2}\tilde{F}^\mu \cdot \tilde{F}_\mu - A_\mu \cdot J^\mu \quad \to \quad \boxed{\varepsilon^{\mu\nu\rho} D_\nu \tilde{F}_\rho = J^\mu}$$

Bianchi identity: $D_\mu \left(\varepsilon^{\mu\nu\rho} D_\nu \tilde{F}_\rho\right) = \tilde{F}^\rho \times \tilde{F}_\rho \equiv 0$

Consistency requires $D_\mu J^\mu = 0$. Three ways to achieve it are:

1. $D_\mu J^\mu(\phi) = 0 \quad \Leftarrow \quad \delta I[A,\phi]/\delta\phi = 0$ [Matter field equation]
2. $D_\mu J^\mu(A) \equiv 0 \quad \Leftrightarrow \quad J^\mu(A) = \delta I_J[A]/\delta A_\mu$. [Higher derivatives]
3. Third Way.

The third way is illustrated by the following choice:

- $\boxed{J^\mu \propto \varepsilon^{\mu\nu\rho} \tilde{F}_\nu \times \tilde{F}_\rho}$

Third Way 3D Yang-Mills

Modified 3D YM with mass-parameter m:

$$\boxed{\varepsilon^{\mu\nu\rho} D_\nu \tilde{F}_\rho + \frac{1}{2m} \underbrace{\varepsilon^{\mu\nu\rho} \tilde{F}_\nu \times \tilde{F}_\rho}_{J^\mu} = 0} \qquad (*)$$

$$D_\mu J^\mu = 2\left(\varepsilon^{\mu\nu\rho} D_\mu \tilde{F}_\nu\right) \times \tilde{F}_\rho \quad \not\equiv 0 \quad \Rightarrow \quad J^\mu \neq \frac{\delta I_J[A]}{\delta A_\mu}$$

However, we can use $(*)$ to get

$$D_\mu J^\mu = -\frac{1}{m} \varepsilon^{\mu\nu\rho} \left(\tilde{F}_\mu \times \tilde{F}_\nu\right) \times \tilde{F}_\rho \quad \equiv 0 \qquad \text{(Jacobi identity)}$$

So $(*)$ is (*third-way*) consistent

➸ $\nexists \quad I_J[A]$. So there isn't an action?

The Action

First order YM action: $L_{YM} = G_\mu \cdot \tilde{F}^\mu - \frac{1}{2} G_\mu \cdot G^\mu$ (G is auxiliary)

Modified Lagrangian

$$L = L_{YM} + \frac{1}{2m}\varepsilon^{\mu\nu\rho}\left(G_\mu \cdot D_\nu G_\rho + \frac{1}{3m} G_\mu \cdot G_\nu \times G_\rho\right)$$

For arbitrary variations $\delta A, \delta G$:

$$\delta L = \delta G_\mu \cdot \underbrace{\underbrace{\left(\tilde{F}^\mu - G^\mu\right)}_{=0 \;\Rightarrow\; G=\tilde{F}}}_{\text{G is still auxiliary!}} + \delta\underbrace{\left(A_\mu + \frac{1}{m}G_\mu\right)}_{A'_\mu} \cdot \underbrace{\left[\varepsilon^{\mu\nu\rho}\left(D_\nu G_\rho + \frac{1}{2m} G_\nu \times G_\rho\right)\right]}_{=0 \;\&\; G=\tilde{F} \quad\Rightarrow (*)}$$

➳ G can be eliminated from the field equations but **not** from the action. There is no action for A alone!

The Massive Chern-Simons Action

In terms of A and $A' = A + m^{-1}G$, the action is

$$I[A, A'] = m\left(I_{CS}[A] - I_{CS}[A']\right) - \frac{1}{2}m^2 \int d^3x\, |A - A'|^2$$

- Mass term breaks $SU(2) \times SU(2) \rightarrow SU(2)_{\text{diag.}}$.
- Parity preserving since $A \leftrightarrow A'$ under parity.

Similar modification of Topologically-Massive YM with mass μ yields

$$I[A, A'] = mI_{CS}[A] - (m-\mu)I_{CS}[A'] - \frac{1}{2}m(m-\mu) \int d^3x\, |A - A'|^2$$

Now parity is broken.

Topologically Massive Gravity

TMG field equation: $G_{\mu\nu} - \frac{1}{\mu} C_{\mu\nu} = 0$, where $C_{\mu\nu}$ is the Cotton tensor:

$$C_{\mu\nu} = \epsilon_\mu{}^{\rho\sigma} D_\rho S_{\sigma\nu} \,, \qquad \left(S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right)$$

$C_{\mu\nu}$ is symmetric, traceless, and satisfies Bianchi identity $D^\mu C_{\mu\nu} \equiv 0$.

Linearization about Minkowski vacuum $\equiv$ $\sqrt{\mathrm{Fierz-Pauli}}$

One spin-2 mode of mass μ.

The TMG action is [Deser, Jackiw, Templeton]

$$I_{TMG}[g] = -I_{EH}[g] + \frac{1}{\mu} I_{CS}[\Gamma(g)]$$

The Chern-Simons term breaks parity.

Third way for 3D massive gravity

Because of Bianchi identities

$$G_{\mu\nu} - \frac{1}{\mu} C_{\mu\nu} = T_{\mu\nu} \quad \Rightarrow \quad \boxed{D_\mu T^{\mu\nu} = 0}$$

Three ways to satisfy consistency condition:

1. $T_{\mu\nu} = T_{\mu\nu}(\phi)$. [matter stress tensor]
2. $D_\mu T^{\mu\nu}(g) \equiv 0 \quad \Leftrightarrow \quad I_{TMG}[g] \to I_{TMG}[g] + I_T[g]$
3. Third Way: $T^{\mu\nu} \propto \boxed{J^{\mu\nu} := \epsilon^{\mu\rho\sigma} \epsilon^{\nu\tau\eta} S_{\rho\tau} S_{\sigma\eta}}$

$$D_\mu J^{\mu\nu} = 2\, \epsilon^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} \quad \neq 0$$

➣ How is this consistent?!

Minimal Massive Gravity

MMG field equation : $G_{\mu\nu} - \frac{1}{\mu} C_{\mu\nu} - \frac{\gamma}{\mu^2} J_{\mu\nu} = 0$ $\quad (\gamma = \text{const.})$

Third Way Consistency

MMG equation: $\Rightarrow$ $C_{\mu\nu} = ()g_{\mu\nu} + ()S_{\mu\nu} + ()S^2_{\mu\nu}$

Use this and symmetry of $S^n_{\mu\nu}$, to get

$$D_\mu J^{\mu\nu} \propto \varepsilon^{\nu\rho\sigma} \left[()S_{\rho\sigma} + ()S^2_{\rho\sigma} + ()S^3_{\rho\sigma} \right] \equiv 0$$

Hence consistency.

Consistency is Third Way type $\Rightarrow$ $\nexists\ I[g]$.

➽ But what about action with auxiliary fields?

MMG action

Einstein-Cartan formulation of 3D GR

Fields: 3-vector one-forms e (dreibein) and ω (dual Lorentz connection).

Field strengths 2-forms: $\begin{cases} \text{Torsion}: & T = de + \omega \times e \\ \text{Curvature}: & R = d\omega + \frac{1}{2}\omega \times \omega \end{cases}$

Lagrangian 3-form: $L_{EC} = e \cdot R$

For TMG we need Lagrange multiplier 3-vector one-form h:

$$L_{TMG} = e \cdot R + h \cdot T + \frac{1}{\mu} L_{CS}(\omega) \qquad [Grumiller, Jackiw, Johansson]$$

For MMG we add the only other even-parity term of right dimension, with new dimensionless parameter γ:

$$L_{MMG} = L_{TMG} + f(\gamma)\, e \cdot h \times h$$

MMG and unitarity in AdS_3

For AdS vacuum of radius ℓ, the semi-classical limit for TMG is

$$\frac{M\ell}{\hbar} \to \infty\,, \quad \frac{\mu}{M} \to 0 \qquad (M^{-1} = \text{ 3D Newton constant})$$

Dimensionless parameter of semi-classical TMG is $\mu\ell/\hbar$.
MMG has the additional dimensionless parameter γ.

Semi-classical unitarity

- Bulk : Spin-2 particle is neither a tachyon nor a ghost
- AdS boundary: Positive central charges $c_\pm$ of asymptotic $\mathrm{Vir}_+ \oplus \mathrm{Vir}_-$ symmetry algebra. [Brown, Henneaux]

- TMG suffers from a Bulk/Boundary clash $\Rightarrow$ non-unitary.
- MMG unitary in region of parameter space where $\alpha \neq 0$.

New Massive Gravity

NMG field equation: $G_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0$ [Bergshoeff, Hohm, PKT]

$$K_{\mu\nu} \propto \frac{\delta}{\delta g_{\mu\nu}} \left[\int d^3x \sqrt{-\det g}\, G^{\mu\nu} S_{\mu\nu} \right]$$

➽ 4th-order in derivatives but only 2nd-order in time derivatives. Linearization about Minkowski vacuum $\rightarrow$ parity doublet of spin-2 modes of mass m. Equivalent to Fierz-Pauli

Lagrangian 3-form, with additional auxiliary Lorentz-vector 1-form f, is

$$L_{NMG} = e \cdot R + h \cdot T - \frac{1}{m^2} \left[f \cdot R + \frac{1}{2} e \cdot f \times f \right]$$

Hamiltonian formulation

For N Lorentz-vector one-forms $e^r = \{e, \omega, h, f, \dots\}$

$$L = \frac{1}{2} g_{rs} a^r \cdot da^s + \frac{1}{6} f_{rst}\, a^r \cdot a^s \times a^t \quad = \quad \frac{1}{2}\varepsilon^{ij} g_{rs} a_i^r \cdot a_j^s + a_0^r \cdot \phi_r$$

$\Rightarrow$ $3N$ primary constraints ϕ_r. By construction, 6 are 1st–class.

Secondary constraints? Dirac's procedure $\Rightarrow e_0 = 0 \;\Rightarrow$ **non-invertible** e.

Assuming **invertible** e $\;\rightarrow\;$ $N-2$ "secondary" constraints.

➽ Upshot is phase-space of dimension $2(N-2)$:

- $N = 2$: CS gravity (3D GR) [Achúcarro, PKT ; Witten]
- $N = 3$: CS-like (one massive spin-2 mode): TMG, MMG
- $N = 4$: CS-like (two massive spin-2 modes): NMG, ...

Exotic Massive Gravity

EMG field equation: $\underbrace{\Lambda g_{\mu\nu}}_{\text{optional CC}} + G_{\mu\nu} - \frac{1}{m^2}H_{\mu\nu} + \frac{1}{m^4}L_{\mu\nu} = 0 \qquad (\dagger)$

$$H_{\mu\nu} = \epsilon_\mu{}^{\rho\sigma} D_\rho C_{\nu\sigma} \,, \qquad \left[D^\mu H_{\mu\nu} \equiv -\epsilon_\nu{}^{\rho\sigma} C_\rho{}^\lambda G_{\sigma\lambda}\right]$$

$$L_{\mu\nu} = \frac{1}{2}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\lambda\tau} C_{\rho\lambda} C_{\sigma\tau} \,, \qquad \left[D^\mu L_{\mu\nu} \equiv -\epsilon_\nu{}^{\rho\sigma} C_\rho{}^\lambda H_{\sigma\lambda}\right]$$

➽ $H_{\mu\nu}\Big|_{\text{linearized}} = \frac{1}{2}K_{\mu\nu}\Big|_{\text{linearized}}$ & $L_{\mu\nu}\Big|_{\text{linearized}} = 0$

$$\Rightarrow \boxed{EMG\Big|_{\text{linearized}} = NMG\Big|_{\text{linearized}}}$$

Different non-linear extension of linearized NMG equations!

➽ Consistency?

EMG Cont'd: Consistency and Action

$$D^\mu \left[H_{\mu\nu} - \tfrac{1}{m^2} L_{\mu\nu}\right] = -\epsilon_\nu{}^{\rho\sigma} C_\rho{}^\lambda \left[G_{\sigma\lambda} - \tfrac{1}{m^2} H_{\sigma\lambda}\right] \quad \neq 0$$

$$= \tfrac{1}{m^4} \epsilon_\nu{}^{\rho\sigma} C_\rho{}^\lambda L_{\sigma\lambda} \quad \text{using EMG equation } (\dagger)$$

$$\propto \epsilon_\nu{}^{\rho\sigma} C^3_{\rho\sigma} \qquad \equiv 0$$

Is there a CS-like action?

Yes, if $\Lambda \neq 0$, but it is parity odd! [cf Witten's "exotic" 3D GR]

$$\mathrm{Limit}_{m^2\to\infty}\ L_{EMG}/m^2 = \underbrace{h \cdot T + L_{LCS}(\omega)}_{\textit{Conformal 3D Gravity}}$$

$\Rightarrow$ EMG is a massive extension of 3D conformal gravity.

- Analogous general construction yields MMG, EMG, ...

Higher Dimensions?

All known examples of the "third way" are in 3D but the idea is, in principle, applicable in any dimension.

Are there new third-way consistent gravitational field equations for $D > 3$?

Lovelock's theorem assumes existence of action $I[g]$. What can be proved without this assumption?

Work in progress with Mehmet Ozkan and Yi Pang: so far, only one half-example, for $D > 3$.