Gravitational perturbation theory from Ambitwistor strings

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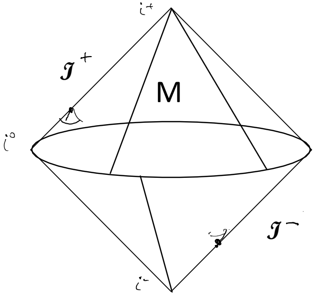
PoToR 5, Palac Wojanow

With David Skinner. arxiv:1311.2564, and T Adamo, E Casali, Y Geyer, A Lipstein, R Monteiro, K Roehrig, & P Tourkine, ...

Based also on Cachazo, He, Yuan arxiv:1306.2962,...

Based on Witten's twistor-string 2003.

- Classical scattering problems.
- Gravitational wave signatures.
- Cosmological perturbations.
- A first approach to Quantum gravity (de Witt . . .).

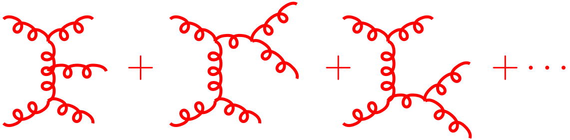


However, space-time approach is extremely complicated:

Yang-Mills/QCD is easier:

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:

$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

+ 19 more pages.

- Computationally intractable.
- Theoretically inscrutable.

Result of a brute force calculation:

$$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$$

$$+ \text{ 19 pages } = \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}\,\delta^4\left(\sum_i k_i\right)$$

[Parke, Taylor 1984].

Hidden structure in general tree amplitudes

Parke-Taylor extended above formula to n particles MHV

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \dots \langle n-1\, n\rangle\langle n\, 1\rangle}\delta^4\left(\sum_i k_i\right)$$

- Interpreted in twistor space [Nair 1986].
- Full YM tree S-matrix, [Witten, Roiban, Spradlin, Volovich 2004].
- Gravity MHV n-point in 4d [Berends, Giele and Kuijf 1986], determinant formula [Hodges 2012], full tree S-matrix [Cachazo, Skinner 2012].
- Full YM and gravity tree S-matrix, all dims [Cachazo, He, Yuan 2013].

Amplitudes: Scatter n plane waves $a_{i\mu}$ for Yang-Mills and $g_{i\mu\nu}$ for gravity, $i = 1, \dots n$

$$a_{i\mu} = \epsilon_{i\mu} e^{ik_i\cdot x}\,, \qquad g_{i\mu\nu} = \epsilon_{i\mu}\epsilon_{i\nu} e^{ik_i\cdot x}\,, \qquad k_i^2 = 0.$$

Polarization data $\epsilon_{i\mu}$ satisfy $k\cdot\epsilon = 0$, $\epsilon \sim \epsilon + \alpha k$. Amplitude is

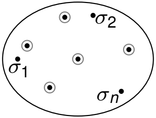
$$\mathcal{M}(1, \dots, n) = \mathcal{M}(k_1, \epsilon_1, \dots, k_n, \epsilon_n).$$

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1$$



.

- Solve for $\sigma_i \in \mathbb{CP}^1$ with the n *scattering equations* [Fairlie 1972]

$$\mathrm{Res}_{\sigma_i}\left(P^2\right) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

$$\Rightarrow \; P^2 = 0 \; \forall \sigma.$$

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Roiban, Spradlin, Volovich, Witten 2004].

Amplitude formulae for massless theories.

Proposition (Cachazo, He, Yuan 2013,2014)

Massless tree amplitudes in d-dims are integrals/residue sums:

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right) \int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i \bar{\delta}(k_i \cdot P(\sigma_i))}{\text{Vol SL}(2,\mathbb{C}) \times \mathbb{C}^3}$$

where $\mathcal{I}^{l/r} = \mathcal{I}^{l/r}(\epsilon_i^{l/r}, k_i, \sigma_i)$ *depend on the theory.*

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l \otimes \epsilon_i^r$ for spin-2 ($k_i \cdot \epsilon_i = 0 \dots$).
- Introduce skew $2n \times 2n$ matrices $M = \begin{pmatrix} A & C \\ -C^t & B \end{pmatrix}$,

  $$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}, , \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad C_{ij} = \frac{k_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad \text{for } i \neq j$$

  and $A_{ii} = B_{ii} = 0$, $C_{ii} = \epsilon_i \cdot P(\sigma_i)$.
- For YM, $\mathcal{I}^l = Pf'(M)$, $\mathcal{I}^r = \prod_i \frac{1}{\sigma_i - \sigma_{i-1}}$.
- For GR $\mathcal{I}^l = Pf'(M^l)$, $\mathcal{I}^r = Pf'(M^r)$.

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More CHY formulae:

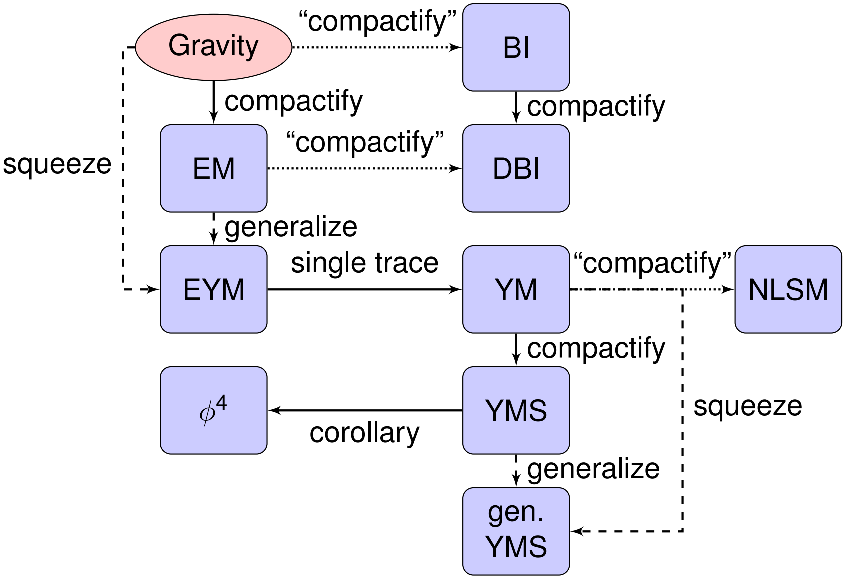
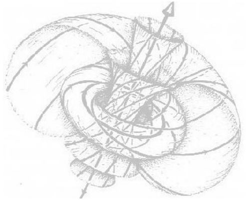


Figure: Theories studied by CHY and operations relating them.

Ambitwistors

Ambitwistor spaces: spaces of complex null geodesics.



- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .

Ambitwistor Strings:

- Twistor-string for $N = 4$ Yang-Mills [Witten, Roiban, Spradlin, Volovich, 2003/4].
- $N = 8$ supergravity [Cachazo, Geyer, Skinner, M., 2012], [Skinner, 2013]
- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- New models for Einstein-YM, DBI, BI, NLS, etc. [Casali, Geyer, M., Monteiro, Roehrig 1506.08771].
- Loop integrands from the Riemann sphere [Geyer, M., Monteiro, Tourkine, 1507.00321, 1511.06315, 1607.08887].

Provide string theories at $\alpha' = 0$ for field theory amplitudes.

Ambitwistors from chiral bosonic strings at $\alpha' = 0$

Bosonic ambitwistor string action:

- Σ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time (M, g), coords $X \in \mathbb{C}^d$, g hol.
- $(X, P) : \Sigma \to T^*M$, $\quad P \in K$, holomorphic 1-forms on Σ.

$$S_B = \int_\Sigma P_\mu \bar{\partial} X^\mu - e\, P^2/2\,.$$

Underlying geometry:

- Lagrange multiplier e enforces $P^2 = 0$,
- e is also worldsheet gauge field for Hamiltonian flow of P^2:

$$\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha).$$

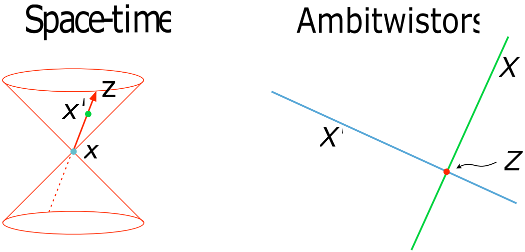
Target reduces to

$$\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}.$$

This is *Ambitwistor space*, space of complexified light rays.
It is holomorphic symplectic with potential $\theta = P_\mu dX^\mu$.

Ambitwistor space $\mathbb{A}$ is space of complexified light rays.

- Light rays primary, an event $x \leftrightarrow$ its lightcone $X \subset \mathbb{A}$.
- Space-time $M =$ space of such $X \subset \mathbb{A}$.



Space-time geometry is encoded in complex structure of $\mathbb{A}$.

Theorem (LeBrun 1983 following Penrose 1976)

Complex structure of $\mathbb{A}$ determines $(M, [g])$. Correspondence stable under deformations of $P\mathbb{A}$ that preserve $\theta = P_\mu \mathrm{d}X^\mu$.

Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \to T^*M$,

$$S_B = \int_\Sigma P_\mu(\bar\partial + \tilde{e}\partial)X^\mu - e\,P^2/2\,.$$

- Gauge fix $\tilde{e} = e = 0$, $\rightsquigarrow$ ghosts & BRST Q
- Introduce vertex operators $V_i \leftrightarrow$ field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathcal{M}_n = \langle V_1 \ldots V_n \rangle$$

For gravity add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_\Psi = \int_\Sigma \Psi_\mu \bar\partial \Psi^\mu + \chi P \cdot \Psi\,.$$

From deformations of $\mathbb{A}$ to the scattering equations

Gravitons $\leftrightarrow$ vertex operators $V_i =$ def'm of action $\delta S = \int_\Sigma \delta\theta$.

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)$.

Proposition

For perturbation $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ of flat space-time

$$\delta\theta = \bar\delta(k \cdot P)\mathrm{e}^{ik\cdot X}(\epsilon \cdot P)^2$$

Proof: Penrose transform.

Ambitwistor repn $\Rightarrow \bar\delta(k \cdot P) \Rightarrow$ scattering equs.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X + 2\pi\sum_i ik\cdot X(\sigma_i)\,.$$

- Gives field equations $\bar\partial X = 0$ and,

$$\bar\partial P = 2\pi\sum_i ik\delta^2(\sigma-\sigma_i)\,.$$

- Solutions $X(\sigma)=X=\text{const.}$, $P(\sigma)=\sum_i \frac{k_i}{\sigma-\sigma_i}d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right)\int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i(\epsilon_i\cdot P(\sigma_i))^2\bar\delta(k_i\cdot P)}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $\sim\int_M R+R^3$, cf [azevedo, Engelund, Hohm, Segal, Zwieback].

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- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holmorphic CFTs.
- Take
$$S = S_B + S^l + S^r$$
where S^l, S^r are some worldsheet matter CFTs.
- Total vertex operators given by
$$v^l v^r \bar{\delta}(k \cdot P)\, \mathrm{e}^{ik\cdot X}$$
with v^l, v^r worldsheet currents from S^l, S^r resp..
- Amplitudes become
$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right) \int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i{}' \bar{\delta}(k_i \cdot P)}{\text{Vol Gauge}}$$
where $\mathcal{I}^l$, $\mathcal{I}^r$ are worldsheet correlators of v^ls, v^rs resp..
- Q-invariance and discrete symmetries (GSO) rule out unwanted vertex operators in good situations.

Worldsheet matter models

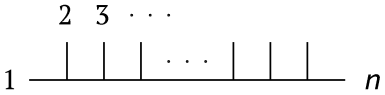
- **Worldsheet SUSY:** $S_\Psi = \int g_{\mu\nu}\Psi^\mu \bar\partial \Psi^\nu - \chi P_\mu \Psi^\mu$ gives CHY Pfaffians from worldsheet correlator

$$\mathcal{I}^{l/r} = \langle u_1 u_2 v_3 \dots v_n \rangle = Pf'(M)\,.$$

- **Free fermions and current algebras:** gives 'Parke-Taylor' correlators + unwanted multi-trace terms

$$\langle v_1 \dots v_n \rangle = \frac{\mathrm{tr}(t_1 \dots t_n)}{\sigma_{12}\sigma_{23} \cdots \sigma_{n1}} + \dots \qquad \text{where} \quad \sigma_{ij} = \sigma_i - \sigma_j.$$

- **Comb system:** [Casali-Skinner]
  Combines level zero current algebra & spin $3/2$ gauging.
  Gives Parke Taylor *without* unwanted multitrace terms, and colour structure as *comb*:



Above lead essentially to original models & formulae:

- $(S^l, S^r) = (S_{\tilde{\Psi}}, S_\Psi) \rightsquigarrow$ type II gravity,
- $(S^l, S^r) = (S_{CS}, S_\Psi) \rightsquigarrow$ heterotic with YM,
- $(S^l, S^r) = (S_{CS}, S_{CS}) \rightsquigarrow$ bi-adjoint scalar.

The latter two come with unphysical gravity.

S_{CS} improves on current algebras in avoiding multi-trace terms and all models critical in 10d.

$$S_{\Psi_1,\Psi_2} = S_{\Psi_1} + S_{\Psi_2}$$

two worldsheet susy's for S^l or S^r (this is maximum).

$$S_{\Psi,\rho} = S_\Psi + S_\rho$$

combines 'real' Fermions with susy.

$$S_{\Psi,CS} = S_\Psi + S_{CS}$$

but with same spin 3/2 gauge field (worldsheet Rarita-Schwinger) in both.

GSO now reverses signs of all fields in matter system.

Ambitwistor strings with combinations of matter

CGMMRS 1506.08771

S^l \ S^r	S_Ψ	S_{Ψ_1,Ψ_2}	$S_{\rho,\Psi}^{(\tilde{m})}$	$S_{CS,\Psi}^{(\tilde{N})}$	$S_{CS}^{(\tilde{N})}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S_{\rho,\Psi}^{(m)}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m \times \mathrm{U}(1)^{\tilde{m}}$		
$S_{CS,\Psi}^{(N)}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*EYMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	
$S_{CS}^{(N)}$	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*gen. YMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	*Biadjoint Scalar* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$

Table: Theories arising from the different choices of matter models.

Models from different geometric realizations of $\mathbb{A}$

Can start with other formulations of null superparticles

- Pure spinor version (Berkovits) $S = \int P \cdot \bar\partial X + p_\alpha \bar\partial \theta^\alpha + \dots$.
- In $d = 4$ have (super) Twistor space $\mathbb{T} := \mathbb{C}^{4|\mathcal{N}}$

$$\mathbb{A} = T^*\mathbb{PT} := \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma W \cdot \bar\partial Z + a\, Z \cdot W$$

$\rightsquigarrow$ Twistor-strings [Witten, Berkovits & Skinner].

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

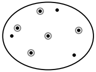
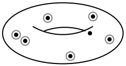
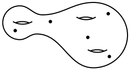
$$S = \int_\Sigma Z \cdot \bar\partial W - W \cdot \bar\partial Z + aZ \cdot W$$

Not twistor string: $(Z, W) \in K^{1/2}$ gives simpler 4d formulae with no moduli. Nonchiral, working with no supersymmetry.

- Adapts to null infinity $\mathscr{I}$: $\mathbb{A} = T^*\mathscr{I}$, admits BMS symmetries and Ward identity proof of soft theorems.

Loops

The string paradigm gives

$\mathcal{M}_n =$  $+$  $+ \ldots +$  $+ \ldots$

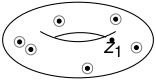
Can we make sense of this at 1 loop, i.e., on a torus?

Need critical model with all anomalies cancelling, i.e., type II super-gravity.

1-loop: the scattering equations on a torus

[Adamo, Casali, Skinner 2013, Casali Tourkine 2014 Geyer,M., Monteiro, Tourkine 2015

On torus $\Sigma_q = \mathbb{C}/\{\mathbb{Z}\oplus\mathbb{Z}\tau\}$, $q = e^{2\pi i\tau}$, solve



$$\bar\partial P = 2\pi i \sum_i k_i \bar\delta(z - z_i) dz \qquad \text{with}$$

$$P = 2\pi i\, \ell dz + \sum_i k_i \left(\frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} + \frac{\theta_1'(z_i - z_0)}{\theta_1(z_i - z_0)} \right) dz\,.$$

zero-modes $\ell \in \mathbb{R}^d \leftrightarrow$ loop momenta (z_0 some fixed basepoint).

Scattering eqs:

$$\mathrm{Res}_{z_i} P^2 := k_i \cdot P(z_i) = 0, \quad i = 2, \ldots, n, \qquad P(z_0)^2 = 0\,.$$

Gives amplitude formula

$$\mathcal{M}^{(1)}_{\mathrm{SG}} = \int \mathcal{I}_q\, d^d\ell\, d\tau\, \bar\delta(P^2(z_0)) \prod_{i=2}^{n} \bar\delta(k_i \cdot P(z_i)) dz_i\,.$$

Localizes on discrete set of solutions to scattering eqs.

With $\mathcal{I}_q = 1$, conjectured to be permutations sum of n-gons.

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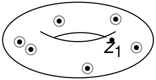

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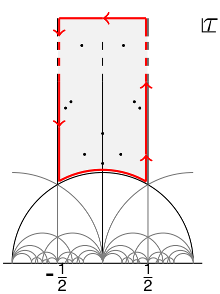
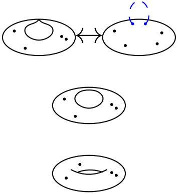
From the elliptic curve to the Riemann sphere

[Geyer, M., Monteiro, Tourkine 1507.00321]





$\sum$\{residues at $P^2(z_0) = 0$\} = \{residue at $q = 0$\} so

$$
\begin{aligned}
\mathcal{M}_n^{(1)} &= \int \mathcal{I}_q \, d^d\ell \, \frac{dq}{q} \, \bar{\partial} \frac{1}{P^2(z_0)} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) dz_i \, , \\
&= - \int \mathcal{I}_0 \, d^d\ell \, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} \, ,
\end{aligned}
$$

Off-shell scattering eqs and 1-loop amplitudes

Reduces to off-shell scattering equations nodal Riemann sphere, i.e., $\mathbb{CP}^1$ with two extra marked points.

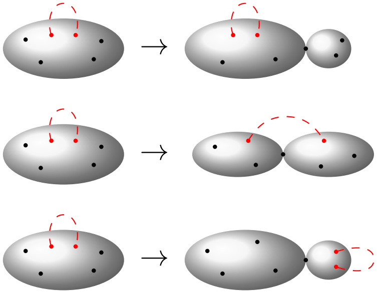
- A priori for 10d type II SUGRA but can be adapted to YM and GR or SUGRA with varying amounts of SUSY.
- Embodies 'double copy', Gravity $=$ (Yang-Mills)^2 at 1-loop.
- One loop formulae on nodal $\mathbb{CP}^1$ is proved by 'Q-cuts':

Q-cuts arise from poles from worldsheet degenerations:

Pole $1/K_I^2$

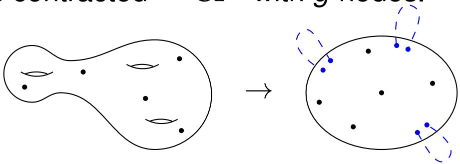
Q-cuts:

As $\ell \to \infty$:



Liouville thm gives identification with Q-cuts [G, M, M, T 1511.06315].

Residue thms localize genus g moduli integrals to bdy cpt with the g a-cycles contracted $\rightsquigarrow \mathbb{CP}^1$ with g nodes.



Fixes g moduli, remaining $2g-3 \leftrightarrow 2g$ new marked points.

- **Proposal:** all-loop integrand is

$$\mathcal{M}_n^{(g)} = \int_{(\mathbb{CP}^1)^{n+2g}} d^{dg}\ell \, \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\mathrm{Vol}\, G} \prod_{r=1}^{g} \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \bar{\delta}(\mathrm{Res}_{\sigma_i} S(\sigma_i)) d\sigma_i \,,$$

$$\text{where } \mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \end{cases} .$$

  Here the PT factors run through each loop.

- **Proved at two loops:** [Geyer, M, Monteiro, Tourkine 1607.08887], now with complete analytic understanding [Geyer, Monteiro 1805.05344].

In flat space we quotient by constraints:

$$\mathcal{G}_0 = P_\mu \Psi^\mu \,, \qquad \widetilde{\mathcal{G}}_0 = \eta^{\mu\nu} P_\mu \widetilde{\Psi}_\mu \,, \qquad \mathcal{H}_0 = \{\mathcal{G}_0, \widetilde{\mathcal{G}}_0\} = \eta^{\mu\nu} P_\mu P_\nu$$

Worldsheet model remains free in curved space, but:

$$\mathcal{G} = P_\mu \Psi^\mu + \partial(\Psi^\mu \Gamma^\nu_{\mu\nu}) + \ldots \,, \quad \widetilde{\mathcal{G}} = g(X)^{\mu\nu} \widetilde{\Psi}_\mu (P_\nu - \Gamma^\lambda_{\nu\kappa} \widetilde{\Psi}_\lambda \Psi^\kappa) + \ldots \,,$$

where $\ldots =$ terms in dilaton and B-field.

Proposition (Adamo, Casali, Skinner, Nekovar 2014, 2018)

The field equations of NS gravity $\Leftrightarrow$ vanishing of anomaly in $\mathcal{G} \circ \widetilde{\mathcal{G}}$ so that

$$\mathcal{H} = \{\mathcal{G}, \widetilde{\mathcal{G}}\} = g(x)^{\mu\nu} P_\mu P_\nu + \ldots \,.$$

This gives concrete amplitude calculations on Plane wave space-times [Adamo, Casali, M., Nekovar 2017].

Ambitwistor strings are far reaching generalizations of twistor-strings:

- Their effective field theory is precisely standard field theory.
- Incorporates double copy Yang-Mills $\leftrightarrow$ gravity ideas.
- Extends to many theories from DBI to Nonlinear σ-models.
- Critical models extend to loops on a Riemann surface.
- Higher genus formulae reduce to $\mathbb{CP}^1$ with *off-shell scattering eqs* $\rightsquigarrow$ loop integrands for non-critical theories.
- Gives usable curved background framework.

Outlook

- Extend curved background to fully nonlinear framework.
- Ambitwistor-string field theory $\rightsquigarrow$ off-shell double copy.
- These structures must play a key role in quantum gravity.

Thank You