The Petrov type D equation on the cross sections of topologically different isolated horizons.

Denis Dobkowski-Ryłko
University of Warsaw

26.09.2018

Wojciech Kamiński, Jerzy Lewandowski,
Adam Szereszewski, Tomasz Pawłowski

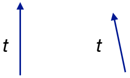
The 5th Conference of the Polish Society on Relativity

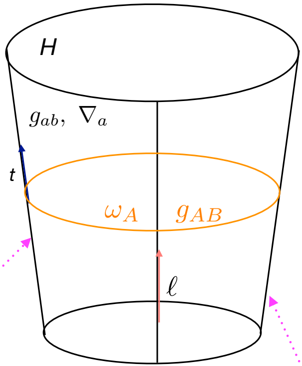
Isolated horizons stationary to the second order

H - **3dim null surface in 4dim spacetime** *M*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

$$\ell^\mu \ell_\mu = 0$$





$$\left.\begin{array}{r} \mathcal{L}_t g_{\mu\nu} = 0 \\ [\mathcal{L}_t, \nabla_\mu] = 0 \\ \mathcal{L}_t R_{\mu\nu\alpha\beta} = 0 \end{array}\right\}$$

Rotation Potential:

$$\nabla_a \ell^b = \omega_a \ell^b$$

Surface Gravity:

$$\kappa^{\ell} = \omega_a \ell^a$$

Non-extremality condition:

$$\kappa^{\ell} \neq 0$$

$$(\omega_A, g_{AB}) \quad \rightarrow \quad (\omega_a, g_{ab}) \quad \rightarrow \quad g_{\mu\nu}\ ,\ \nabla_\mu\ ,\ R_{\mu\nu\alpha\beta}$$

Spacetime null frame adapted to H

- The null frame:
  - $g_{AB} = m_A\bar{m}_B + m_B\bar{m}_A$
  - $\omega^{(\ell)}_A = (\alpha + \bar{\beta})m_A + (\bar{\alpha} + \beta)\bar{m}_A$
- Following N-P notation, denote the directional derivatives by:

$$\delta = m^a\partial_a, \qquad D = \ell^a\partial_a \tag{1}$$

- Spacetime Weyl tensor $C^{\mu}{}_{\alpha\beta\gamma}$ in the null frame formalism may be expressed by the following complex valued N-P components:

$$\begin{aligned} \Psi_0 &= C_{4141}, & \Psi_1 &= C_{4341} & \Psi_2 &= C_{4123}, \\ \Psi_3 &= C_{3432}, & \Psi_4 &= C_{3232}. \end{aligned} \tag{2}$$

- Four components are constant along the null generators of H:

$$D\Psi_I = 0, \qquad I = 0, 1, 2, 3. \tag{3}$$

- Assume:

$$D\Psi_4 = 0 \tag{4}$$

The Weyl tensor

- Due to vanishing of the expansion and shear of ℓ:

$$\Psi_0 = \Psi_1 = 0 \tag{5}$$

- Ψ_2 can be expressed in terms of the Gaussian curvature and scalar invariant $\mathcal{O}$:

$$\begin{aligned} \Psi_2 &= \bar{\delta}\beta - \delta\alpha + (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} \\ &- 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) + \frac{1}{6}\Lambda \\ &= -\frac{1}{2}(K + i\mathcal{O}) + \frac{1}{6}\Lambda \end{aligned} \tag{6}$$

- The component Ψ_3 reads:

$$\begin{aligned} \Psi_3 &= \bar{\delta}\mu - \delta\lambda + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\bar{\beta}) \\ &= \frac{1}{\kappa^{(\ell)}} \left(\bar{\delta} + 3\alpha + 3\bar{\beta}\right) \Psi_2, \end{aligned} \tag{7}$$

The Petrov type D equation

- Using Bianchi identities and the assumption that $D\Psi_4 = 0$ we find that:

$$\Psi_4 = \frac{1}{2\kappa^{(\ell)}} \left(\bar{\delta}\Psi_3 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 \right) \tag{8}$$

- Petrov type D condition:

$$2\Psi_3^2 = 3\Psi_2\Psi_4; \tag{9}$$

- Putting all the pieces together gives:

$$(\bar{\delta} + \alpha - \bar{\beta})\bar{\delta}\Psi_2(x)^{-\frac{1}{3}} \;=\; 0. \tag{10}$$

which can be written as:

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \Psi_2(x)^{-\frac{1}{3}} = 0, \tag{11}$$

Theorem 1. *Suppose H is a 3-dim non-extremal isolated null surface stationary to the second order in a 4-dimensional spacetime such that the vacuum Einstein equations with cosmological constant Λ are satisfied. Then, the necessary and sufficient condition for the spacetime Weyl tensor to be of the Petrov type D at each point of the null geodesic $x \in S$ is, that Ψ_2 satisfies the following two conditions:*

$$\Psi_2(x) \ \neq 0, \tag{12}$$

and

$$\bar{m}^A \bar{m}^B \nabla_B \nabla_A \Psi_2(x)^{-\frac{1}{3}} = 0. \tag{13}$$

Petrov type D solution on 2-sphere for axisymmetric IH

- consider metric tensor:

$$g_{AB}dx^A dx^B = R^2\left(\frac{1}{P^2}dx^2 + P^2 d\varphi^2\right) \tag{14}$$

- rotation scalar invariant $\mathcal{O}$ can be expressed by

$$\mathcal{O} = -\Delta U \tag{15}$$

- with the constraints $\Big($lack of conical singularity of $P(x)$ and $U(x)\Big)$:

$$\begin{aligned} P|_{x=\pm 1} &= 0 \\ \lim_{x\to\pm 1} \partial_x P^2 &= \mp 2 \\ P\partial_x U\,|_{x=\pm 1} &= 0 \end{aligned}$$

Applying these to type D equation we obtain a complex ordinary differential equation for Ψ_2:

$$\frac{P^2}{2R^2}\partial_x^2\Psi_2^{-\frac{1}{3}} = 0 \tag{16}$$

The result of its integration is:

$$\Psi_2 = (c_1 x + c_2)^{-3} \tag{17}$$

Comparing the two expressions for Ψ_2:

$$\Psi_2 = (c_1 x + c_2)^{-3} = -\frac{1}{2}(K - i\Delta U) + \frac{1}{6}\Lambda \tag{18}$$

$$\frac{4R^2}{(c_1 x + c_2)^3} = \partial_x^2 P^2 + 2i\partial_x(P^2\partial_x U) + \frac{2}{3}R^2\Lambda. \tag{19}$$

Solution to type D equation: $c_1 = 0$

Assuming $c_1 = 0$ and integrating eq. (19) twice yields:

$$P^2 = 1 - x^2 \tag{20}$$

and diff. eq. for rotation potential:

$$0 = (1 - x^2)\partial_x U \implies U = \text{const} \tag{21}$$

Additionally, we find that Gaussian curvature is also constant:

$$K = -\frac{1}{2R^2}\partial_x^2 P^2 = \frac{1}{R^2} \tag{22}$$

$\implies$ embeddable in the Schwarzschild-(anti) de Sitter spacetime or the near extremal horizon spacetimes

Solution to type D equation: $c_1 \neq 0$

Integrating eq. (19) twice and applying boundary conditions yields:

- $R^2 = \frac{3\gamma}{\Lambda\gamma - 6}$,
- $P^2 = 1 - x^2 + \frac{6}{6-\gamma\Lambda}\frac{(x^2-1)^2}{x^2+\eta^2}$
- $\partial_x U = \frac{1}{2\eta}\frac{3\eta^4 - x^2 + \eta^2(x^2+1)}{\left(x^2+\eta^2\right)\left(\eta^2+1-\gamma\frac{\Lambda}{6}(x^2+\eta^2)\right)}$
- $\omega^{(\ell)}{}_\varphi = \star dU_\varphi = -P^2\partial_x U = \frac{(1-x^2)\left[3\eta^4+\eta^2+x^2(\eta^2-1)\right]}{2\eta(\gamma\frac{\Lambda}{6}-1)(x^2+\eta^2)^2}$

where $\eta = -i\frac{c_2}{c_1}$ and $\gamma = \frac{(c_2^2 - c_1^2)^2}{c_2}$.

$\Longrightarrow$ embeddable in the Kerr-(anti) de Sitter spacetime or the near extremal horizon spacetimes

We now find the area:

$$A = 4\pi R^2 = 12\pi \frac{\gamma}{\gamma\Lambda - 6}, \tag{23}$$

and the angular momentum using the imaginary part of Ψ_2:

$$\begin{aligned} J &= -\frac{1}{4\pi}\int_S \phi Im\Psi_2\epsilon \\ &= \frac{1}{8(\gamma\frac{\Lambda}{6} - 1)^2}\frac{\gamma}{\eta}. \end{aligned} \tag{24}$$

Function ϕ is defined up to an additive constant as the generator of the vector field $\Phi = \partial_\varphi$ namely: $\phi_{,B} := \Phi^A\epsilon_{AB}$

No hair theorem for axisymmetric solutions to the Petrov type D equation

Theorem 2: *The family of axisymmetric solutions of the Petrov type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by two numbers (A,J): the area and angular momentum, respectively. They can take the following values:*

- for $\Lambda > 0$, $J \in (-\infty, \infty)$ for $A \in \left(0, \frac{12\pi}{\Lambda}\right)$ and $\mid J \mid \in \left[0, \frac{A}{8\pi}\sqrt{\frac{\Lambda A}{12\pi} - 1}\right)$ for $A \in \left(\frac{12\pi}{\Lambda}, \infty\right)$;
- for $\Lambda \leq 0$, $J \in (-\infty, \infty)$ and $A \in (0, \infty)$.

Theorem 3: *Suppose S is a compact 2-surface of genus > 0. The only solutions to the Petrov type D equation with a cosmological constant Λ are (g, ω) such that:*

$$d\omega = 0 \qquad\qquad K = \text{const} \neq \frac{\Lambda}{3} \tag{25}$$

Isolated horizon of the Hopf bundle topology

$$\omega^+(1) = 0 = \omega^-(-1)$$

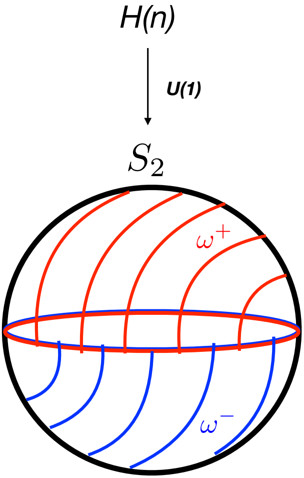
$$d\omega^+ = d\omega^- = \mathcal{O}R^2 dx \wedge d\phi$$

$$\int_S \mathcal{O} dA = 2\pi n\kappa$$

Equation to solve:

$$\frac{4R^2}{c_1 x + c_2} = \partial_x^2 P^2 + 2iR^2\mathcal{O}$$

$$\begin{cases} P|_{x=\pm 1} = 0 \\ \lim_{x\to\pm 1} \partial_x P^2 = \mp 2 \end{cases}$$



(Moncrief 1983)

Isolated horizon of the Hopf bundle topology

$$P^2(x) = \frac{\Big(1 - x^2\Big)\Big(1 + \beta^2(1+\alpha^2) + 2\alpha\beta x\Big)}{\beta^2(1+\alpha^2) + 2\alpha\beta x + x^2}$$

$$\mathcal{O} = Im\left[\frac{-i\Big(1 - \beta^2(\alpha+i)^2\Big)^2}{\beta\Big(x + \beta(\alpha+i)\Big)^3}\right]$$

where α and β are related with c_1 and c_2 in the following way:

$$\frac{c_2}{c_1} = \beta(\alpha + i)$$

$$\alpha = -\frac{1}{4}n\kappa$$

Isolated horizon of the Hopf bundle topology

$$\omega^{+} = \left(Im\left[\frac{i\Big(1-\beta^2(\alpha+i)^2\Big)^2}{2\beta\Big(x+\beta(\alpha+i)\Big)^2} \right] + \frac{-1+2\alpha\beta+\beta^2(1-\alpha^2)}{2\beta} \right) d\phi$$

$$\omega^{-} = \left(Im\left[\frac{i\Big(1-\beta^2(\alpha+i)^2\Big)^2}{2\beta\Big(x+\beta(\alpha+i)\Big)^2} \right] + \frac{1+2\alpha\beta-\beta^2(1-\alpha^2)}{2\beta} \right) d\phi$$

Thank you