Palatini frames in scalar-tensor theories of gravity

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Scalar-tensor theories in the Palatini approach

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Why to combine these two approaches?

- introduction of new class of theories, reducing to metric S-T in some cases;
- possible cosmological application, e.g. analysis of inflation (Racioppi 2017, Rasanen and Wahlman 2017);
- no comprehensive study and general formalism developed so far.

Action functional

The action functional we postulate is the following:

$$\begin{aligned} S[g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}, \Phi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} \Big[& \mathcal{A}(\Phi) R(g, \Gamma) \\ & - \mathcal{B}(\Phi) g^{\mu\nu} \nabla_{\mu}\Phi \nabla_{\nu}\Phi - A_1^{\mu}(g, \Gamma) \mathcal{C}_1(\Phi) \nabla_{\mu}\Phi \\ & - A_2^{\mu}(g, \Gamma) \mathcal{C}_2(\Phi) \nabla_{\mu}\Phi - \mathcal{V}(\Phi) \Big] + S_{\text{matter}}[e^{2\alpha(\Phi)} g_{\mu\nu}, \chi], \end{aligned} \tag{1}$$

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where:

$$A_1^\mu(g, \Gamma) = g^{\mu\nu} g^{\alpha\beta} \nabla_\nu g_{\alpha\beta} = g^{\mu\nu} g^{\alpha\beta} Q_{\nu\alpha\beta}, \tag{2a}$$

$$A_2^\mu(g, \Gamma) = -g^{\mu\nu} g^{\alpha\beta} \nabla_\alpha g_{\nu\beta} = -g^{\mu\nu} g^{\alpha\beta} Q_{\alpha\nu\beta}. \tag{2b}$$

Causal and affine structure decoupled $\rightarrow$ **metric tensor and affine connection transform independently under conformal change**

Transformation group and self-consitency conditions

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We introduce the following transformations:

$$\bar{g}_{\mu\nu} = e^{2\gamma_1(\Phi)} g_{\mu\nu}, \tag{3a}$$

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + 2\delta^{\alpha}_{(\mu}\partial_{\nu)}\gamma_2(\Phi) - g_{\mu\nu}g^{\alpha\beta}\partial_{\beta}\gamma_3(\Phi), \tag{3b}$$

$$\bar{\Phi} = f(\Phi) \tag{3c}$$

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- $\gamma_3 = 0 \rightarrow$ geodesic transformation
- $\gamma_2 = \gamma_3 \rightarrow$ Weyl transformation

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$$\bar{\mathcal{A}}(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\mathcal{A}(\check{f}(\bar{\Phi})), \tag{4a}$$

$$\begin{aligned}\bar{\mathcal{B}}(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\Big[&\mathcal{B}(\check{f}(\bar{\Phi}))(\check{f}'(\bar{\Phi}))^2 + (n-1)\Big(n\mathcal{A}(\check{f}(\bar{\Phi}))\check{\gamma}_2'(\bar{\Phi})\check{\gamma}_3'(\bar{\Phi}) - \mathcal{A}(\check{f}(\bar{\Phi}))\left(\check{\gamma}_2'(\bar{\Phi})\right)^2 \\ &- \mathcal{A}(\check{f}(\bar{\Phi}))\left(\check{\gamma}_3'(\bar{\Phi})\right)^2 - \frac{d\mathcal{A}(\check{f}(\bar{\Phi}))}{d\bar{\Phi}}(\check{\gamma}_2'(\bar{\Phi}) + \check{\gamma}_3'(\bar{\Phi})) \\ &- (n-2)\mathcal{A}(\check{f}(\bar{\Phi}))\check{\gamma}_1'(\bar{\Phi})(\check{\gamma}_2'(\bar{\Phi}) + \check{\gamma}_3'(\bar{\Phi}))\Big) \\ &+ \check{f}'(\bar{\Phi})\Big(\mathcal{C}_1(\check{f}(\bar{\Phi}))((n-2)n\check{\gamma}_1'(\bar{\Phi}) - 2(n+1)\check{\gamma}_2'(\bar{\Phi}) + 2\check{\gamma}_3'(\bar{\Phi})) \\ &- \mathcal{C}_2(\check{f}(\bar{\Phi}))((n-2)\check{\gamma}_1'(\bar{\Phi}) - (n+3)\check{\gamma}_2'(\bar{\Phi}) + (n+1)\check{\gamma}_3'(\bar{\Phi}))\Big)\Big],\end{aligned} \tag{4b}$$

$$\bar{\mathcal{C}}_1(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\left[\check{f}'(\bar{\Phi})\mathcal{C}_1(\check{f}(\bar{\Phi})) - \mathcal{A}(\check{f}(\bar{\Phi}))\left(\frac{n-1}{2}\check{\gamma}_2'(\bar{\Phi}) + \frac{n-3}{2}\check{\gamma}_3'(\bar{\Phi})\right)\right], \tag{4c}$$

$$\bar{\mathcal{C}}_2(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\left[\check{f}'(\bar{\Phi})\mathcal{C}_2(\check{f}(\bar{\Phi})) - \mathcal{A}(\check{f}(\bar{\Phi}))\left((n-1)\check{\gamma}_2'(\bar{\Phi}) - \check{\gamma}_3'(\bar{\Phi})\right)\right], \tag{4d}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = e^{n\check{\gamma}_1(\bar{\Phi})}\mathcal{V}(\check{f}(\bar{\Phi})), \tag{4e}$$

$$\bar{\alpha}(\bar{\Phi}) = \alpha(\check{f}(\bar{\Phi})) + \check{\gamma}_1(\bar{\Phi}). \tag{4f}$$

Transformation group and self-consitency conditions

Transformations: $\mathfrak{A} \to \mathfrak{B}$ with (γ_i, f) and $\mathfrak{B} \to \mathfrak{C}$ with $(\bar{\gamma}_i, \bar{f})$

Composition: $\mathfrak{A} \to \mathfrak{C}$ with $(\bar{\bar{\gamma}}_i, \bar{\bar{f}})$, where

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In other dimensions - group reduced to two different subgroups:
a)$\gamma_1 = 0$ with arbitrary γ_2, γ_3; contains Weyl transformations ($\gamma_2 = \gamma_3$) and the projective transformations ($\gamma_3 = 0$) of the connection;
b)$\gamma_2 + \gamma_3 =$ const with arbitrary conformal transformation of the metric tensor.

Invariant quantities

Invariants are quantities built from the functions $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{V}, \alpha\}$ such that their **functional dependence on them is the same in every frame**; also, their **value at a given spacetime point remains unchanged**.

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Integral invariant:

$$\mathcal{I}_E^n(\Phi) = \int \left(\pm \frac{\mathcal{A}(\Phi)\mathcal{B}(\Phi) + \mathcal{A}'(\Phi)[\mathcal{C}_2(\Phi) - n\mathcal{C}_1(\Phi)]}{\mathcal{A}(\Phi)^2} \pm \frac{(n^2-5)\mathcal{C}_2(\Phi)^2 - 4\mathcal{C}_1(\Phi)^2 + 2(4+n-n^2)\mathcal{C}_1(\Phi)\mathcal{C}_2(\Phi))}{(n-2)(n-1)\mathcal{A}(\Phi)^2} \right)^{\frac{1}{2}} d\Phi \tag{5}$$

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$\mathcal{I}_E^n$ invariant iff:

1. $n = 4$;
2. $\gamma_2' + \gamma_3' = 0$;
3. $\gamma_1(\Phi) = \frac{1}{n-2}\ln(\mathcal{A}(\Phi)$.

Definition

The **Einstein frame in the Palatini theory** is characterized by specific values of four out of six arbitrary functions $\{\mathcal{A}, \dots, \alpha\}$:
$\mathcal{A} = 1$, $\mathcal{B} = \epsilon_{\text{Palatini}}$, $\mathcal{C}_1 = \mathcal{C}_2 = 0$.
The action functional is given by:

$$S[g^E_{\mu\nu}, (\Gamma^E)^\alpha_{\mu\nu}, \Phi] =$$
$$\frac{1}{2\kappa^2} \int_\Omega d^n x \sqrt{-g^E} \Big(R(g^E, \Gamma^E) - \epsilon_{\text{Palatini}} (g^E)^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \mathcal{V}(\Phi) \Big)$$
$$+ S_{\text{matter}} \left[e^{2\alpha(\Phi)} g^E_{\mu\nu}, \chi \right],$$

where $\epsilon_{\text{Palatini}} \equiv (\pm 1, 0)$ is a three valued function.

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The quantity $\mathcal{I}_E^n$ is invariant for all frames relatable to the Einstein frame.

Scalar-tensor extension of Palatini $F(\hat{R})$-gravity

Consider the action of minimally coupled $F(\hat{R})$-gravity

$$S_F(g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}) = \frac{1}{2\kappa^2} \int_{\Omega} \mathrm{d}^n x \sqrt{-g} F(\hat{R}) + S_{\text{matter}}(g_{\mu\nu}, \chi) \tag{6}$$

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In the scalar-tensor representation:

$$S(g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}, \Phi) = \frac{1}{2\kappa^2}\int_{\Omega} \mathrm{d}^n x \sqrt{-g}\left(\Phi\hat{R} - U_F(\Phi)\right) + S_{\text{matter}}(g_{\mu\nu}, \chi) \tag{7}$$

where: $U_F(\Phi) \equiv \hat{R}(\Phi)\Phi - F(\hat{R}(\Phi))$ and $\Phi = \frac{dF}{d\hat{R}}$

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where: $U_F(\Phi) \equiv \hat{R}(\Phi)\Phi - F(\hat{R}(\Phi))$ and $\Phi = \frac{dF}{d\hat{R}}$

It can be written in the Einstein frame by setting: $g^E_{\mu\nu} = \Phi^{\frac{2}{n-2}} g_{\mu\nu}$

$$S_{EP}(g^E_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}, \Phi) = \frac{1}{2\kappa^2}\int_{\Omega} \mathrm{d}^n x\sqrt{-g^E}\left(\hat{R} - \bar{U}_F(\Phi)\right) + S_{\text{matter}}(\Phi^{-\frac{2}{n-2}} g^E_{\mu\nu}, \chi) \tag{8}$$

with $\bar{U}_F = \frac{U_F}{\Phi^{\frac{n}{n-2}}}$

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In order to find out whether a given S-T theory in the Palatini approach arises from some $F(\hat{R})$, one needs to check if it is equivalent to (9).

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In order to find out whether a given S-T theory in the Palatini approach arises from some $F(\hat{R})$, one needs to check if it is equivalent to (9).

This occurs when the following conditions are satisfied:

- $\bar{\mathcal{A}} = 1$ or $\bar{\mathcal{C}}_2 = n\bar{\mathcal{C}}_1$;
- $\mathcal{I}^n_E = 0$

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2. invariants retain their properties only for frames located on the same orbit;
3. Einstein frame, with $\mathcal{A} = 1$ and $\mathcal{C}_1 = \mathcal{C}_2 = 0$, is fully analogous to the metric frames;
4. Palatini S-T theores are the minimal extension of the Palatini $F(\hat{R})$-gravity; an arbitrary S-T theory is equivalent to some $F(\hat{R})$ theory if the coefficients $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{V}, \alpha\}$ satisfy well-defined conditions.