POISSON-RIEMANNIAN GEOMETRY

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Quantum spacetime hypothesis:

Quantum Gravity??? $\Rightarrow$ Quantum Riemannian geometry (NCRG) $\xrightarrow[\lambda_{Planck}\to 0]{}$ Classical Riem. Geom.

This is not surjective, not every classical geometry is `quantisable'!
(… Einstein's equation etc some kind of quantisability constraint?)

① Quantum Riemannian geometry on any algebra $(A, \Omega, \mathrm{d}, g, \nabla^{\text{QLC}}, \dots)$

LTCC lectures 2011   (a bimodule approach — DVM) & Book w/ Beggs 2019

② Semiclassicalisation $O(\lambda_{Planck})$: classical-quantum gravity

$(C^\infty(M), \omega, g, \nabla)$ ω (Poisson) tensor g metric ∇ (flat) Poisson conn

quantises functions   quantises diff. calc.

w/ Beggs, J. Geom. Phys. 114 (2017)

I. Quantum differentials on an algebra A

Classically, $C^\infty(M) = \Omega^0(M) \subset \Omega(M) = \oplus_i \Omega^i(M)$

Ω^1 space of 1-forms, e.g. `differentials' $\quad \mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i$

$f\mathrm{d}g = (\mathrm{d}g)f \in \Omega^1$

$$\wedge : \Omega \otimes_A \Omega \to \Omega, \quad \mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega)\wedge \eta + (-1)^{|\omega|}\omega \wedge \mathrm{d}\eta$$

$\omega \wedge \eta = (-1)^{|\omega||\eta|}\eta \wedge \omega, \quad \mathrm{d}^2 = 0$ `graded Leibniz rule'

- algebra A over k we drop the (graded) commutativity, just keep:

Ω^1	a((db)c)=(a(db))c	`bimodule'
$\mathrm{d} : A \to \Omega^1$	d(ab)=(da)b+a(db)	`Leibniz rule'
$\{\sum a\mathrm{d}b\} = \Omega^1$		`surjectivity'
$\ker \mathrm{d} = k.1$		(`connected')

- require this to extend to a DGA $\Omega = T_A\Omega^1/\mathcal{I} = \oplus_n \Omega^n, \quad \mathrm{d}^2 = 0$
- inner if exists $\theta \in \Omega^1, \quad \mathrm{d} = [\theta, \ \}$

Nice problem: take your favourite algebra and classify all differential structures (perhaps with some symmetry)

Thm w/ Tao Pac. J. Math (2016)

bicovariant surjective

$$\Omega^1(U(\mathfrak{g})) \longleftrightarrow \zeta \in Z^1(\mathfrak{g}, \Lambda^1)$$

$$\mathrm{d}x = 1 \otimes \zeta(x), \ \Omega^1 = U(\mathfrak{g}) \otimes \Lambda^1$$

e.g. bicovariant connected classical dim $\longleftrightarrow$ pre-Lie algebra

$$\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \qquad [x, y] = x \circ y - y \circ x$$

$$(x \circ y) \circ z - (y \circ x) \circ z = x \circ (y \circ z) - y \circ (x \circ z)$$

$$\Lambda^1 = \{\mathrm{d}y \mid y \in \mathfrak{g}\} \cong \mathfrak{g} \quad [x, \mathrm{d}y] = \mathrm{d}(x \circ y)$$

$\Rightarrow \ \Omega(U(\mathfrak{g}))$ by skew-symmetrisation of products of Λ^1

- Example $\mathfrak{g} : [x^i, t] = \lambda x^i$ (bicrossproduct model/`kappa' Mink)
- Example $\mathfrak{g} = \mathrm{Vect}(M)$ and torsion free flat connection

$$x \circ y = \nabla_x y, \quad \nabla_{[x,y]} z = \nabla_x \nabla_y z - \nabla_y \nabla_x z$$

$$A = U(\mathrm{diff}(M)) \qquad [x, y] = \nabla_x y - \nabla_y x$$

- Example $\mathfrak{g} = V, \quad [\ ,\ ] = 0, \quad (V, \circ)$ comm associative algebra

e.g. $V = C^\infty(M) \qquad A = C_{\rm poly}(V) \qquad [f, \mathrm{d}g] = \mathrm{d}(fg)$

e.g. $V = \mathbb{C}.x, \quad x \circ x = \lambda x, \quad A = \mathbb{C}[x], \quad [x, \mathrm{d}x] = \lambda \mathrm{d}x$

=> $$\mathrm{d}f(x) = \frac{f(x) - f(x - \lambda x)}{\lambda}\mathrm{d}x$$

$$\mathrm{d}x^2 = (\mathrm{d}x)x + x\mathrm{d}x = 2x\mathrm{d}x - \lambda\mathrm{d}x = (2x - \lambda)\mathrm{d}x$$

Propn X discrete set $\Omega^1(C(X)) \quad \longleftrightarrow$ directed graphs on X

$\Omega^1 = \mathrm{span}_k\{\omega_{x\to y}\}$

$$f.\omega_{x\to y} = f(x)\omega_{x\to y}, \omega_{x\to y}.f = f(y)\omega_{x\to y} \qquad \mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}$$

If a graph is bidirected, define

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_{C(X)} \omega_{y\to x} \qquad g_{x\to y} \in k$$

`metric lengths'

Quantum metrics

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes_A \mathrm{d}x^\nu$$

$$g \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \wedge(g) = 0$$ `quantum symmetric'

invertible in the sense exists inverse: $(\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

$$((\ ,\ )\otimes \mathrm{id})(\omega \otimes g) = \omega = (\mathrm{id} \otimes (\ ,\ ))(g \otimes \omega), \quad \forall \omega \in \Omega^1$$

$$a(\omega, \eta) = (a\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta a)$$ `bimodule map (tensorial)'

need this to be able to contract/ `raise/lower' via metric, eg to have well defined contraction:

$$(\ ,\ ) \otimes \mathrm{id} : \quad \Omega^1 \underset{A}{\otimes} \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1$$ " $T_{\mu\nu\rho} \mapsto g^{\mu\nu} T_{\mu\nu\rho}$ "

but

$$(\omega, g^1)g^2 = \omega \qquad g = g^1 \underset{A}{\otimes} g^2$$

$$\Longrightarrow (\omega, g^1)g^2 a = \omega a = (\omega a, g^1)g^2 = (\omega, a g^1)g^2$$

$$\Longrightarrow \qquad ag = ga, \quad \forall a \in A$$ need metric to be central

Connections and curvature

Classically, a connection assigns a covariant derivative

$$\nabla \mathrm{d}x^\mu = -\Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\nu \otimes_A \mathrm{d}x^\rho$$ (Christoffel symbols)

Similarly for any differential algebra $(A, \Omega^1, \mathrm{d})$

bimodule connection: $\nabla : \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \sigma : \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$

$$\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega \qquad \nabla(\omega f) = \sigma(\omega \otimes \mathrm{d}f) + (\nabla\omega)f$$

(Quillen, Karoubi,...) (Michor, Dubois-Violette, ...)

such connections extend to tensor products

$$\omega \otimes \eta \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta)$$

more generally $\nabla_E : E \to \Omega^1 \otimes_A E, \quad \sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$

${}_A\mathcal{E}_A = \{(E, \nabla_E, \sigma_E)\}$ is a monoidal category by $\otimes_A$

`metric compatible' now makes sense $\nabla g = 0$ but is quadratic

- torsion free also makes sense $T_\nabla : \Omega^1 \to \Omega^2$ $T_\nabla = \wedge\nabla - \mathrm{d}$

- quantum Levi-Civita connection (QLC) $T_\nabla = \nabla g = 0$

- Curvature

$$R_\nabla : \Omega^1 \to \Omega^2 \underset{A}{\otimes} \Omega^1 \qquad R_\nabla = (\mathrm{d} \underset{A}{\otimes} \mathrm{id} - (\wedge \underset{A}{\otimes} \mathrm{id})(\mathrm{id} \underset{A}{\otimes} \nabla))\nabla$$

Lemma: (1st Bianchi identity)

$$\wedge(R_\nabla) = \mathrm{d} \circ T_\nabla - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes T_\nabla)\nabla$$

- Laplacian $\Delta : A \to A, \quad \Delta = (\, , \,)\nabla \mathrm{d}$

- *-compatibility in *-algebra case

$$[\mathrm{d}, *] = 0, \quad g^\dagger = g, \quad \sigma \dagger \nabla * = \nabla; \quad \dagger = \mathrm{flip}(* \otimes *)$$

Example 1

$[x^i, t] = \lambda x^i \qquad [t, \mathrm{d}x^i] = -\lambda \mathrm{d}x^i, \qquad [t, \mathrm{d}t] = \lambda\alpha \mathrm{d}t$

Phys Rev D (2015) (w/ Tao)

$$g = \sum_{i,j}^{n-1} a_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + \sum_{i}^{n-1} b_i(\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) + c\mathrm{d}t \otimes \mathrm{d}t$$

$[f, g] = 0, \forall f$ => a_{ij}, b_i, c of degree $-2, \alpha - 1, 2\alpha$

add spherical symmetry => unique form of quantum metric

$$g = \delta^{-1}\mathrm{d}\Omega^2 + ar^{-2}\mathrm{d}r \otimes \mathrm{d}r + br^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + cr^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t$$

$$\bar{\delta} = \frac{c\alpha^2}{b^2 - ac} \qquad a, b, c \in \mathbb{R},\ \delta > 0 \qquad b^2 - ac > 0$$

$$G = -\frac{(n-2)(n-3)}{2}\delta g + ((n-3)\delta - \bar{\delta})\mathrm{d}\Omega^2$$

- solves Einst Eqn with Maxwell field and cosmological constant

$$F = q\sqrt{b^2 - ac}\, r^{\alpha-1}\mathrm{d}t \wedge \mathrm{d}r \qquad \Lambda = \frac{(n-2)(n-3)}{2}\delta - q^2 G_N, \qquad q^2 G_N = \frac{1}{2}\left((n-3)\delta - \bar{\delta}\right)$$

- This is the Bertotti-Robinson metric. We are forced to it!

Example 2 different calculus

Class. Quant. Gravity 31 (2014) (w/ Beggs)

$$[x^i, \mathrm{d}t] = \lambda\beta\mathrm{d}x^i, \quad [t, \mathrm{d}x^i] = \lambda(\beta-1)\mathrm{d}x^i, \quad [t, \mathrm{d}t] = \lambda\beta\mathrm{d}t$$

$\beta = 1, n = 2$

=> unique form of quantum metric

$$g = \mathrm{d}r\otimes\mathrm{d}r + b(v^*\otimes v + \lambda(\mathrm{d}r\otimes v - v^*\otimes\mathrm{d}r))$$

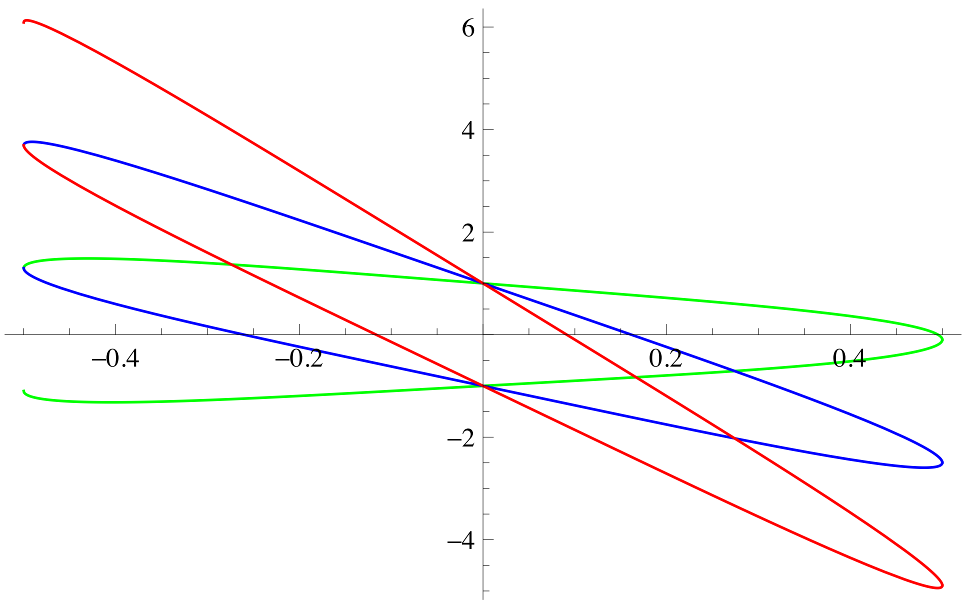
$b \in \mathbb{R}$

$b \neq 0$

$$v = r\mathrm{d}t - t\mathrm{d}r, \quad v^* = (\mathrm{d}t)r - t\mathrm{d}r$$

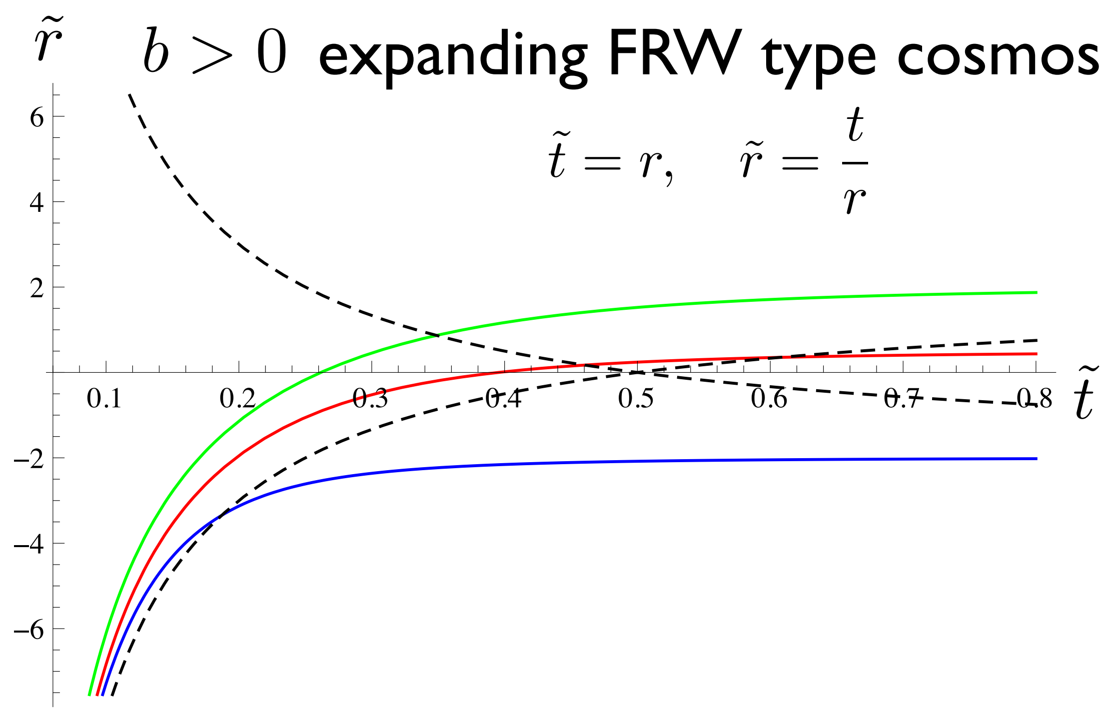
=> in classical limit

$b < 0$ **strong grav source at r=0**



$b > 0$ **expanding FRW type cosmos**

$$\tilde{t} = r, \quad \tilde{r} = \frac{t}{r}$$



$$g = -\mathrm{d}\tilde{t}^2 + R(\tilde{t})^2\mathrm{d}\tilde{r}^2, \quad R(\tilde{t}) = \sqrt{b}\tilde{t}^2$$

2. Poisson-Riemannian Geometry

w/Beggs J. Geom. Phys. 2017

$A_0 = C^\infty(M)$ quantisation at order λ means a `Poisson' bracket

$$a.b - b.a = \lambda\{a,b\} + O(\lambda^2)$$

$\{\ ,\ \} \leftrightarrow \omega^{ij}$ `Poisson' tensor

Similarly, quantization of $\Omega^1(M)$ at order λ implies new physical field:

$$a.\mathrm{d}b - (\mathrm{d}b).a = \lambda \nabla_{\hat{a}} \mathrm{d}b + O(\lambda^2)$$

$\hat{a} = \{a,\ \}$

$\Longrightarrow$ ∇ a Poisson pre-connection along Hamiltonian vec. fields

$$\nabla_{\hat{a}}(b\mathrm{d}c) = \{a,b\}\mathrm{d}c + b\nabla_{\hat{a}}\mathrm{d}c \qquad \mathrm{d}\{a,b\} = \nabla_{\hat{a}}\mathrm{d}b - \nabla_{\hat{b}}\mathrm{d}a$$

- At order λ^2 the bimodule associativity is $(\nabla_{\hat{a}}\nabla_{\hat{b}} - \nabla_{\hat{b}}\nabla_{\hat{a}} - \nabla_{\widehat{\{a,b\}}})\mathrm{d}c = 0$

  $$(\text{just consider } [a,[b,\mathrm{d}c]] + [b,[\mathrm{d}c,a]] + [\mathrm{d}c,[a,b]] = 0)$$

  non-flat connection => nonassociativity at $O(\lambda^2)$ not at order λ

- Equiv to Lie Rinehart connection [Huebschmann '90] also called `contravariant connection' [Hawkins] $\quad \nabla_{\hat{a}} = \nabla_{\mathrm{d}a}$

- Convenient to suppose ∇ an actual connection restricting to $\nabla_{\hat{a}}$

Simplified Axioms of PRG:

$\widehat{\nabla}$ l.c. of classical g

(1) $\nabla g = 0$ <=> $\widehat{\nabla} = \nabla + S$ $\qquad S^a_{bc} = \frac{1}{2} g^{ad}(T_{dbc} - T_{bcd} - T_{cbd})$

(2) $(\widehat{\nabla}_k \omega)^{ij} + \omega^{ir} S^j_{rk} - \omega^{jr} S^i_{rk} = 0$ <=> Poisson compatibility

(3) $\widehat{\nabla}_\rho \mathcal{R}_{\mu\nu} + S^\beta{}_{\alpha\nu} H^\alpha{}_{\beta\rho\mu} - S^\beta{}_{\alpha\mu} H^\alpha{}_{\beta\rho\nu} = 0$ `QLC condition'

where $H^\alpha{}_{\beta\mu\nu} = g_{\beta\gamma}\omega^{\gamma\rho}(\nabla_\rho S^\alpha{}_{\mu\nu} + R^\alpha{}_{\nu\mu\rho})$ $\qquad \mathcal{R}_{\mu\nu} = \frac{1}{2}(H^\alpha{}_{\alpha\mu\nu} - H^\alpha{}_{\alpha\nu\mu})$

Theorem: can then quantise to order λ

(1) => `quant metric' $g_1 := q^{-1}(g - \frac{\lambda}{4} g_{ij}\omega^{is}(T^j_{nm;s} - R^j{}_{nms} + R^j{}_{mns})\mathrm{d}x^m \otimes_0 \mathrm{d}x^n)$

(2) => `quant exterior algebra' $\wedge_1 = \wedge + \lambda(\cdots), \mathrm{d}_1 = \mathrm{d}$

(3) => `quantum levi-civita conn' $\widehat{\nabla}_1 = \widehat{\nabla} + \lambda(\cdots), \ \sigma_1 = \mathrm{flip} + \lambda(\cdots)$

Proof is functorial, any classical v. bundle with connection (E, ∇_E) gets quantised

$Q(E) = E$ but with deformed product $\forall a \in A, \ e \in E$

$$a \bullet e = a\,e + \frac{\lambda}{2}\omega^{ij} a_{,i}(\nabla_{Ej} e) + O(\lambda^2) \qquad e \bullet a = a\,e - \frac{\lambda}{2}\omega^{ij} a_{,i}(\nabla_{Ej} e) + O(\lambda^2)$$

… etc

Failure ω Jacobi identity => quantum algebra nonassociative at $O(\lambda^2)$

Failure ∇ flat => quantum differential forms nonassociative at $O(\lambda^2)$

Theorem: the geometric Laplacian $\Delta = (\ ,\ ) \circ \widehat{\nabla} \circ \mathrm{d}$ gets deformed

$$\Delta_1 f = \Delta f + \frac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{;\alpha})(\widehat{\nabla}_{\beta}\mathrm{d}f)_{\gamma}$$

w/Fritz Class. Qua. Grav (2017)

and similarly the manifold dimension gets deformed to

$$\dim_1 := (,)_1(g_1) = \dim(M) + \frac{\lambda}{2}\{g_{\mu\nu}, g^{\mu\nu}\}$$

Similarly, given a lifting

$$i_1(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \frac{1}{2}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu) + \lambda I^{\mu\nu}_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$$

Define Ricci_1 by lifting and contracting curvature of the QLC, and Ricci scaler by contracting this (using the quantum metric) $S_1 = (\ ,\ )_1\mathrm{Ricci}_1$

I don't know how to formulate and define the Einstein tensor!

Example Sphere $\sum(z^i)^2 = 1$ $\nabla = \widehat{\nabla}$ (the Levi-Civita connection) so $S = 0$

$$\omega = \mathrm{Vol}^{-1}$$

=> $$[z^i, z^j]_\bullet = \lambda\epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j]_\bullet = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n$$

associative algebra U(su2), non associative diff calculus due to curvature

$$\begin{aligned} g_1 =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu - \frac{\lambda}{2(z^3)^2}\mathrm{d}z^3 \otimes_1 \epsilon_{3ij} z^i \mathrm{d}z^j + \lambda\widetilde{\mathrm{Vol}} \\ =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu + \frac{\lambda}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right) \end{aligned}$$

$$\nabla_1 \mathrm{d}z^\mu = -z^\mu \bullet g_1 = -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \lambda z^\mu \widetilde{\mathrm{Vol}} + \frac{\lambda}{2}\left(\mathrm{d}z^3 \otimes_1 (\epsilon^{\mu\beta} g_{\beta\gamma} + \frac{z^\mu z^\beta}{(z^3)^2}\epsilon_{\beta\gamma})\mathrm{d}z^\gamma\right)$$

$$= -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2(z^3)^2}\left(\epsilon_{3ij} z^\mu z^3 \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \epsilon^\mu{}_{\nu 3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^\nu\right)$$

=> $\mathrm{Ricci}_1 = -\frac{1}{2} g_1$ $\quad \Delta_1 = \Delta$ undeformed at first order

Uniqueness theorem

w/Fritz Class. Qua. Grav (2017)

$$g = a^2(r,t)\mathrm{d}t \otimes \mathrm{d}t + b^2(r,t)\mathrm{d}r \otimes \mathrm{d}r + c^2(r,t)(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi)$$

generic spherically symmetric metric & p.b. => unique quantisation to $O(\lambda^2)$

$$S_{022} = c\partial_t c, \quad S_{122} = c\partial_r c, \quad S_{033} = c\partial_t c \sin^2(\theta), \quad S_{133} = c\partial_r c \sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

=> $r,\ t,\ \mathrm{d}r,\ \mathrm{d}t$ central `unquantized radius and time' and at each *r, t*

$$[z^i, z^j] = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j] = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n \qquad \sum (z^i)^2 = 1$$

`non associative fuzzy sphere' as above

E.g. Schwarzschild black hole; $\quad \mathrm{Ric}_1 = O(\lambda^2) \qquad \Delta_1 = \Delta$

Need algebraic independence eg Bertotti-Robinson metric is not generic and has a quantisation as above, with

$$\{r, t\} = r \qquad \nabla \mathrm{d}r = \tfrac{1}{r}\mathrm{d}r \otimes \mathrm{d}r, \quad \nabla \mathrm{d}t = -\tfrac{\alpha}{r}\mathrm{d}t \qquad \Delta_1 = \Delta$$

Example 2D Bicrossproduct model

$$\{r,t\}=r \quad ; \nabla \mathrm{d}r=0 \text{ and } \nabla \mathrm{d}t=r^{-1}(\mathrm{d}t\otimes\mathrm{d}r-\mathrm{d}r\otimes\mathrm{d}t).$$

$v=r\mathrm{d}t-t\mathrm{d}r$ => $\nabla \mathrm{d}r=\nabla v=0$ => $g=\mathrm{d}r\otimes\mathrm{d}r+bv\otimes v$

=> $\widehat{\nabla} v=-\frac{2v}{r}\otimes\mathrm{d}r,\quad \widehat{\nabla}\mathrm{d}r=\frac{2bv}{r}\otimes v.$ $\quad \partial_v f=\frac{1}{r}f_{,t},\quad \partial_r f=f_{,r}+\frac{t}{r}f_{,t}$

=> $\Delta=\frac{2}{r}(\partial_r f)+\partial_r^2 f+b^{-1}\partial_v^2 f$ => 0-modes $\psi_\omega^{\pm}(t,r)=\mathrm{e}^{\imath\frac{\omega t}{r}}\mathrm{e}^{\pm\imath\frac{\omega}{r\sqrt{-b}}}$

=> Δ_1 does get deformed due to contorsion => 0-modes:

$$\psi_\omega^U(t,r)=\mathrm{e}^{\imath\frac{\omega t}{r}}\mathrm{e}^{-\imath\frac{\omega}{r\sqrt{-b}}}\mathrm{e}^{-\frac{\omega}{2r}\lambda_P\left(1+\frac{r}{\omega}\sqrt{-b}\,\tau_U(\frac{\omega}{r\sqrt{-b}})\right)}+O(\lambda_P^2)$$

$$\psi_\omega^M(t,r)=\mathrm{e}^{\imath\frac{\omega t}{r}}\sin(\frac{\omega}{r\sqrt{-b}})\mathrm{e}^{-\frac{\omega}{2r}\lambda_P\left(1+\frac{r}{\omega}\sqrt{-b}\,\tau_M(\frac{\omega}{r\sqrt{-b}})\right)}+O(\lambda_P^2),\quad |r|>\frac{|\omega|}{\pi\sqrt{-b}}$$

$$\tau_M(s)\sim-2\tan(\frac{s}{2}),\quad \tau_U(s)\sim\frac{\pi}{2}\tanh(s)-\imath\ln(e^{-\gamma}+2|s|)$$

$\gamma\approx 0.577$ is the Euler constant

=> should lead to phenomenology ...

Overview

deform $\Omega(M)$

often get quantum anomaly for diff calculus with symmetry
e.g. Beggs/SM Pac. J. Math (2006)

quantum anomaly for diff calculus: PRG eqns with a particular metric and pb, ∇ may have to be curved

(a) work with nonassoc geometry

(b) change the poisson tensor

(c) absorb the anomaly in a higher dimension

$$\mathrm{d}f = \mathrm{d}_{class} f + \frac{\lambda}{2}(\Delta f)\theta'$$

e.g. black hole in wave op approach CMP (2012)

And finally…

If $\lambda = \hbar$ this could apply to quantum mechanics … M=phase sp
If $\lambda = \lambda_P$ this is a new paradigm of semi-classical quantum gravity

Bonus 1 of NCRG: works over any field k e.g.

$\mathbb{F}_2$ `digital geometry' w/ A. Pachol *arXiv:1807.08492 (math.dg)*

Algebra	Relations	dim A	dim Ω^1	# metrics	# QLCs	# $R_\nabla = 0$	# Ricci $= 0$
$\mathbb{F}_2\mathbb{Z}_2$	$x^2 = 0$	2	1	2	1	1	1
$\mathbb{F}_2(\mathbb{Z}_2)$	$x^2 = x$	2	1	1	1	1	1
$\mathbb{F}_4$	$x^2 = 1 + x$	2	1	3	1	1	1
	$x^2 = y^2 = xy = 0$	3	2	0	-	-	-
$\mathbb{F}_2(\mathbb{Z}_3)$	$x^2 = x,\ y^2 = y,\ xy = 0$	3	2	1	4	1	3
	$x^2 = x,\ y^2 = xy = 0$	3	2	0	-	-	-
$\mathbb{F}_2\mathbb{Z}_3$	$x^3 = x + x^2$	3	2	3	12	1	3
	$x^3 = 0$	3	2	0	-	-	-
$\mathbb{F}_8$	$x^3 = 1 + x^2$	3	2	7	40	13	18
	$x^2 = x, y^2 = yx = 0, xy = y$	3	2	0	-	-	-

All digital algebras dim <4 that admit a parallelisable diff calculus and top form degree 2, including 9 which are Ricci flat but not flat

Common phenomena e.g. in dim 3:

(i) $\Delta = 0$ *if and only if* $\underline{\dim} = 0$.

(ii) If $\Delta \neq 0$ *and* $\mathrm{Tr}(\Delta) = 1$ *then* Δ *has one mode with eigenvalue 1 and two with eigenvalue 0*

Bonus 2 on NCRG: includes discrete geometry e.g.

Cayley graph on ad-stable set generators $\mathcal{C}$ of a group X

edges: $x \to xa, a \in \mathcal{C}$

$$e_a f = R_a(f)e_a, \quad \mathrm{d}f = \sum_{a\in\mathcal{C}} \partial^a(f)e_a$$

$$\partial^a = R_a - \mathrm{id}$$

left-invariant 1-forms: $e_a = \sum_{x\in X} \omega_{x\to xa}$

$$\mathrm{d} = [\theta, \ \} \qquad \theta = \sum_{a\in\mathcal{C}} e_a$$

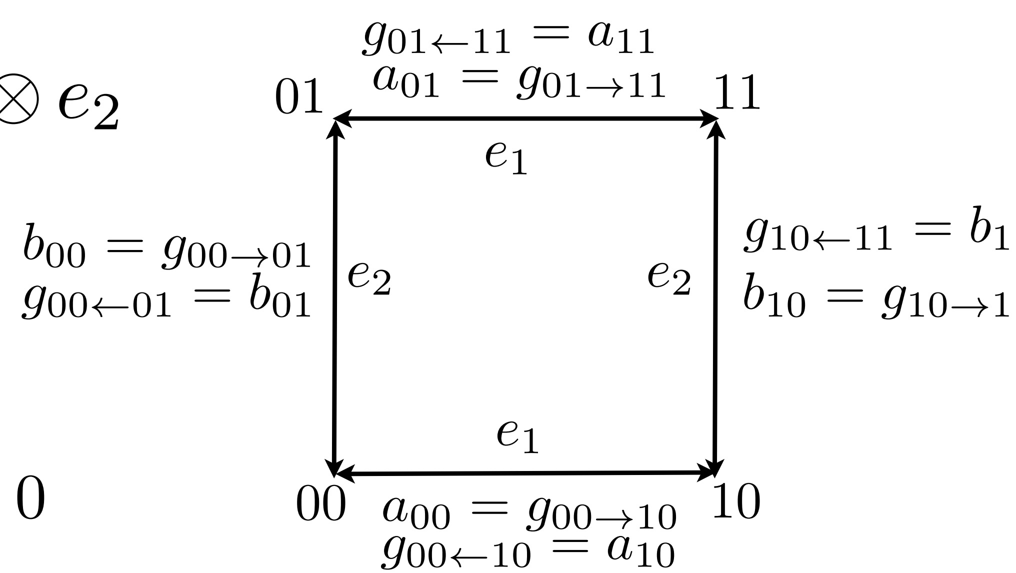
e.g. $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\qquad e_a^2 = 0,\ e_a e_b + e_b e_a = 0 \qquad \mathrm{d}e_a = 0$

Phil. Trans. Roy Soc. (2018)

=> metric $g = ae_1 \otimes e_1 + be_2 \otimes e_2$

for some functions a,b

It is natural to suppose g symmetric `lengths': $\partial^1 a = \partial^2 b = 0$



=> 1-parameter moduli space of QLCs

$$\nabla\omega = \theta\otimes\omega - \sigma(\omega\otimes\theta)$$

$$Q = (q, q^{-1}, q^{-1}, q)$$

$$\sigma = \begin{pmatrix} -Q^{-1} & 0 & 0 & \frac{a(R_1\alpha-1)}{b} \\ 0 & \alpha-1 & \beta & 0 \\ 0 & \alpha & \beta-1 & 0 \\ \frac{b(R_2\beta-1)}{a} & 0 & 0 & Q \end{pmatrix}$$

cf `8-vertex R-matrix'

$$\alpha = (\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}}) \qquad \beta = (1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1)$$

with curvature e.g.

$$R_\nabla e_1 = \Big(Q^{-1}R_1\alpha - Q\alpha + (1-\alpha)(R_1\beta-1) + \frac{R_2 a}{a}(R_2\beta-1)(R_2R_1\alpha-1)\Big)\mathrm{Vol}\otimes e_1$$

$$+\Big(Q^{-1}(1-\alpha) + \alpha(R_2\alpha-1) + Q^{-1}\frac{R_1 b}{a}(\beta^{-1}-1)) + \frac{b}{a}(R_2\beta-1)R_2\beta\Big)\mathrm{Vol}\otimes e_2$$

=> geometric quantum Laplacian

$$\Delta f = (\ ,\ )\nabla(\partial_i f e_i) = -\frac{2}{a}\partial_1 f - \frac{2}{b}\partial_2 f + \partial_i f(\ ,\ )\nabla e_i = \left(\frac{Q^{-1} - R_2\beta}{a}\right)\partial_1 f - \left(\frac{Q + R_1\alpha}{b}\right)\partial_2 f$$

and quantum Ricci scaler curvature for the antisymm lift,

$$S = -\frac{1}{4ab}\left((3+q+(1-q)\chi)\frac{\partial_2 a}{\alpha} + (1-q^{-1}-(3+q^{-1})\chi)\frac{\partial_1 b}{\beta}\right) \qquad \chi = (1,-1,-1,1)$$

Choice of measure $\mu = |ab| = ab$ =>

$$\int S = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu S = (a_{00} - a_{01})^2\left(\frac{1}{a_{00}} + \frac{1}{a_{01}}\right) + (b_{00} - b_{10})^2\left(\frac{1}{b_{00}} + \frac{1}{b_{10}}\right)$$

measures the `energy' in the gravitational field. `bathtub' shape minimised at a, b constant (`rectangular' geometry)

Laplacian has generically 3 massive modes but evals becomes complex for certain background values best seen in momentum mode expansion:

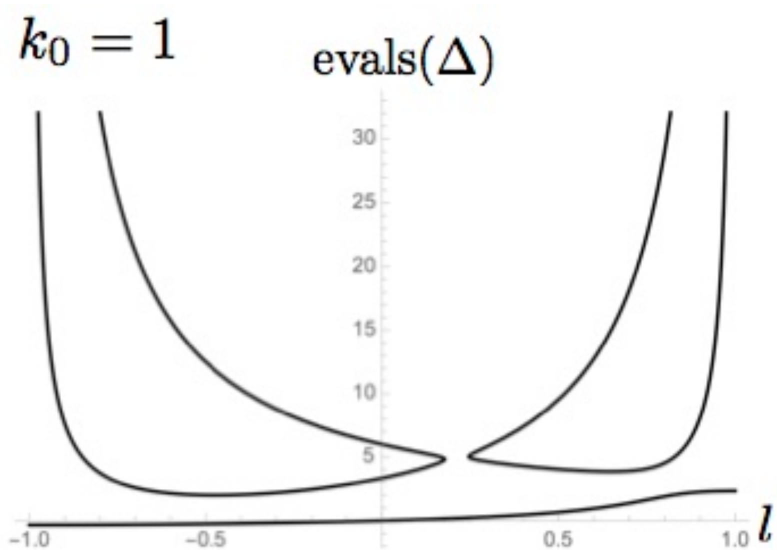
$$\phi(i,j) = (-1)^i = (1,1,-1,-1), \quad \psi(i,j) = (-1)^j = (1,-1,1,-1)$$

$$a = k_0 + k_1\psi, \quad b = l_0 + l_1\phi$$

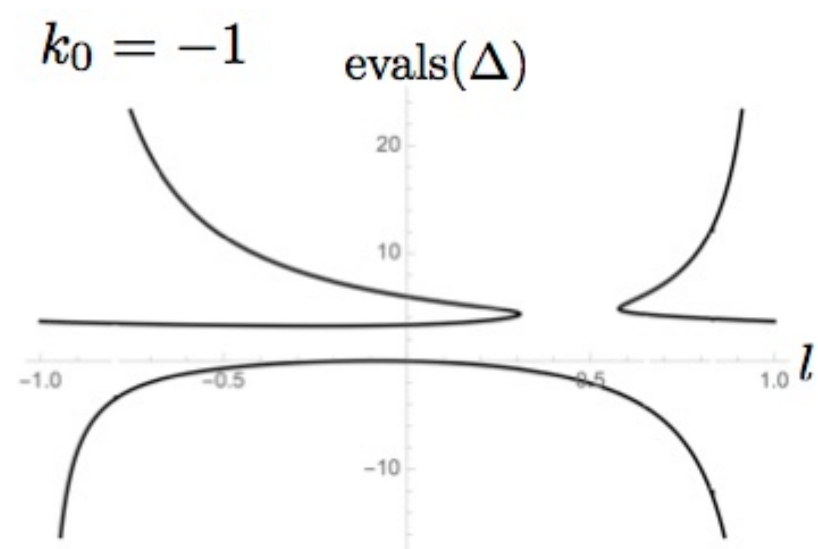
$$\int S = 8\left(\frac{k_0 k^2}{1-k^2} + \frac{l_0 l^2}{1-l^2}\right) \qquad k = k_1/k_0 \qquad l = l_1/l_0$$

$$\int_{-1}^{1}\int_{-1}^{1} \mathrm{d}k\,\mathrm{d}l\, e^{\imath \int S}$$

converges ... finite quantum gravity?



euclidean



minkowski