Construction of quasi-convex foliations with monotonous Geroch mass

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Motivations:

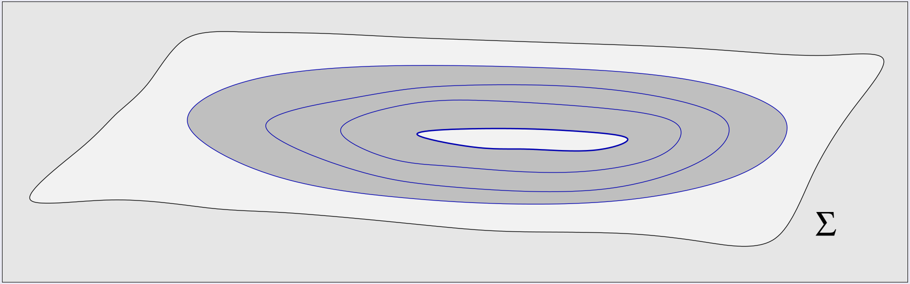
GR is a **metric theory of gravity**:

- it is highly non-trivial to talk about, for instance, the mass, energy or angular momentum of certain bounded spatial regions
- "... it is almost certain that we have to understand conserved (or quasi conserved) quantities which can control the field in a more local manner. In other words, we expect some concept of quasi-local mass will be useful."
- efforts to prove the **positive mass theorem** and the **Penrose inequalities** using certain quasi-local quantities Geroch (1973), Wald, Jang (1977), Jang (1978), Kijowski (1986), Chru?ciel (1986), Jezieski, Kijowski (1987), Huisken, Ilmanen (1997, 2001), Frauendiener (2001), Bray (2001), Malec, Mars, Simon (2002), Bray, Lee (2009),. . .

The aim is to outline:

- a **simple construction** of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that
  - the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
  - the foliation gets to be **quasi-convex** w.r.t. the constructed three-metric

Foliations by topological two-spheres:



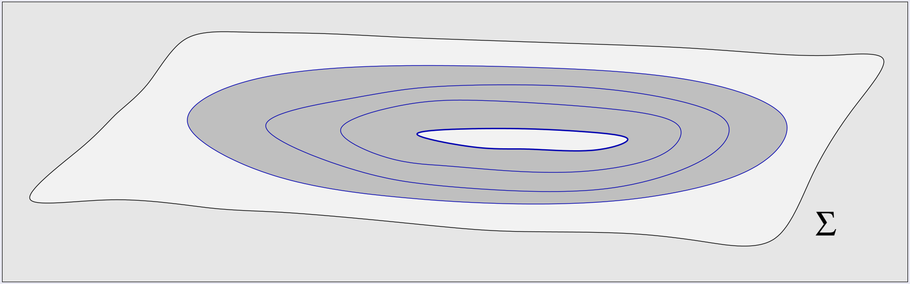
- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume

$$\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$$

  i.e. Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

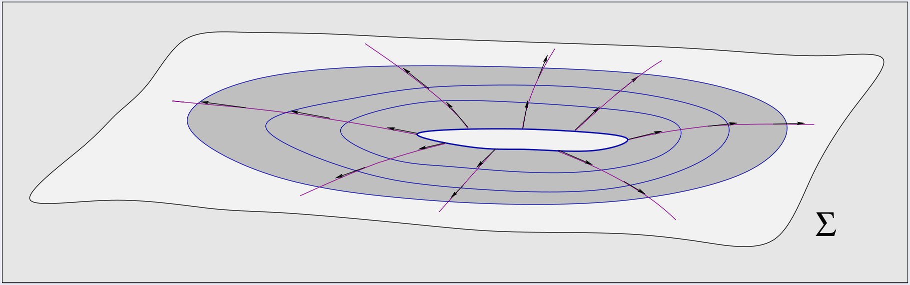
- $$\Longrightarrow \quad \boxed{\partial_i \rho \ \ \& \ \ h^{ij} \ \longrightarrow \ \widehat{n}_i \, , \widehat{n}^i = h^{ij} \widehat{n}_j \ \ \ldots \ \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$$
- '$\widehat{\ }$' to distinguish quantities that could also be viewed as fields on the leaves

Quasi-convex foliations:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$
- **the extrinsic curvature** given by the symmetric tensor field
$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l \,\widehat{n}_j = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij},} \qquad D_i, \mathscr{L}_{\widehat{n}}$$
- a $\rho = const$ level surface is called to be **quasi-convex if its mean curvature**, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i\,\widehat{n}_j}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow**, ("evolution vector field") w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i \partial_i \rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i \widehat{n}_i = (\widehat{n}^i \partial_i \rho)^{-1}} \qquad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

Variation of the area:

- to any quasi-convex foliation $\exists$ a (quasi-local) **orientation of the leaves** $\mathscr{S}_\rho$
- a flow ρ^i is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\epsilon}}$ of the $\rho = const$ level surfaces, w.r.t. ρ^i

$$\boxed{\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \,\widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N}(\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\ \widehat{\boldsymbol{\epsilon}},}$$

  the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol{\epsilon}}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol{\epsilon}} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_{\widehat{N}}\widehat{\gamma}_{ij}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol{\epsilon}}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i\widehat{N}^i$, were applied.
- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$\boxed{h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$

  gets to be singular
- for **quasi-convex foliations** $\boxed{\widehat{N}\widehat{K}^l{}_l > 0}$ $\Longrightarrow$ the **area is increasing** w.r.t. ρ^i
- the orientations by $\widehat{n}^i$ and ρ^i coincide

The Geroch mass:

- the (quasi-local) Geroch mass (equal to the Hawking mass only if $K^i{}_i = 0$)

$$\boxed{m_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for quasi-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$\boxed{W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

- **if both $\mathscr{A}_\rho$ and $W(\rho)$ were non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{m_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation** we shall use **relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$ {}^{(3)}\!R = \widehat{R} - \Big\{ 2\, \mathscr{L}_{\widehat{n}} (\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl} \widehat{K}^{kl} + 2\, \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} \Big\} \qquad (*) $$

$$ \begin{aligned} \mathscr{L}_\rho W &= -\int_{\mathscr{S}_\rho} \mathscr{L}_\rho \Big[\big(\widehat{K}^l{}_l\big)^2 \,\widehat{\boldsymbol\epsilon} \Big] = -\int_{\mathscr{S}_\rho} \Big\{ \widehat{N}\, \mathscr{L}_{\widehat{n}} \Big[\big(\widehat{K}^l{}_l\big)^2 \,\widehat{\boldsymbol\epsilon} \Big] + \mathscr{L}_{\widehat{N}} \Big[\big(\widehat{K}^l{}_l\big)^2 \,\widehat{\boldsymbol\epsilon} \Big] \Big\} \\ &= -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big) \Big[2\, \mathscr{L}_{\widehat{n}} \,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 \Big] \,\widehat{\boldsymbol\epsilon} - \int_{\mathscr{S}_\rho} \widehat{D}_i \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{N}^i \Big] \widehat{\boldsymbol\epsilon} \\ &= -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big) \Big[\big(\widehat{R} - {}^{(3)}\!R \big) - \widehat{K}_{kl} \widehat{K}^{kl} - 2\, \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} \Big] \,\widehat{\boldsymbol\epsilon} \end{aligned} $$

- where on 1^{st} line $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\, \widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\, \widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}}\, \widehat{\boldsymbol\epsilon} = (\widehat{D}_i \widehat{N}^i)\, \widehat{\boldsymbol\epsilon}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{N}^i \Big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\boxed{\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N} = \widehat{D}^l\big(\widehat{N}^{-1}\widehat{D}_l\widehat{N}\big) + \widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})}$$

- and by introducing the trace-free part of $\widehat{K}_{ij}$

$$\boxed{\overset{\circ}{\widehat{K}}_{ij} = \widehat{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,(\widehat{K}^l{}_l), \qquad \widehat{K}_{kl}\widehat{K}^{kl} = \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + \tfrac{1}{2}\,(\widehat{K}^l{}_l)^2}$$

- and using the vanishing of the integral of the total divergence $\widehat{D}^l(\widehat{N}^{-1}\widehat{D}_l\widehat{N})$

$$\boxed{\begin{aligned}\mathscr{L}_\rho W = &-\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}} \\ &+ \int_{\mathscr{S}_\rho}(\widehat{N}\,\widehat{K}^l{}_l)\left[{}^{(3)}\!R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\right]\widehat{\boldsymbol{\epsilon}}\end{aligned}}$$

Rigidity of the setup:

- **if** the product $\boxed{\widehat{N}\widehat{K}^l{}_l}$ **could be replaced** by its **mean value**

- $$\boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \frac{\int_{\mathscr{S}_\rho} \widehat{N}\widehat{K}^l{}_l\,\widehat{\boldsymbol{\epsilon}}}{\int_{\mathscr{S}_\rho}\widehat{\boldsymbol{\epsilon}}}} \qquad \boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}]\cdot\mathscr{L}_\rho\, m_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho \log[\mathscr{A}_\rho])\,W \geq 0}$$

- once in addition to h_{ij} **a foliation and a flow are fixed** not only the **mean curvature** $\widehat{K}^l{}_l$ **BUT** the **lapse** $\widehat{N}$ and the **shift** $\widehat{N}^i$ get also to be fixed

$$\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i\,\widehat{n}_j} \qquad \boxed{\widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1}} \qquad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

- the only "freedom" is a relabeling of the leaves by using a function $\overline{\rho} = \overline{\rho}(\rho)$ but this cannot yield more than a rescaling $\widehat{N} \to \widehat{N}(\mathrm{d}\rho/\mathrm{d}\overline{\rho})$ of the lapse
- **(!)** at best $\boxed{\widehat{N}\widehat{K}^l{}_l}$ is a smooth positive function on the leaves of the foliation

How to get control on the monotonicity?

What we have by hands: $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho : \Sigma \to \mathbb{R},\, \rho^i = (\partial_\rho)^i\}$

- a Riemannian metric h_{ij} defined on a three-surface Σ
- Σ is foliated by topological two-spheres: a function $\rho : \Sigma \to \mathbb{R}$ is chosen
- a flow ρ^i, with $\rho^i \partial_i \rho = 1$, was also fixed on Σ
- in coordinates (ρ, x^A) adapted to the flow $\boxed{\rho^i = (\partial_\rho)^i}$, with components $\delta^i{}_\rho$

$$\boxed{\widehat{N}^i = \delta^i{}_A \widehat{N}^A \qquad \text{and} \qquad \widehat{\gamma}_{ij} = \delta^A{}_i\, \delta^B{}_j \widehat{\gamma}_{AB}}$$

  $\widehat{N}^A$ and $\widehat{\gamma}_{AB}$ depend smoothly on ρ, x^A, where A takes the values $2, 3$

- line element of the Riemannian metric h_{ij}

$$\boxed{\mathrm{d}s^2 = \widehat{N}^2 \mathrm{d}\rho^2 + \widehat{\gamma}_{AB} \left(\mathrm{d}x^A + \widehat{N}^A \mathrm{d}\rho\right) \left(\mathrm{d}x^B + \widehat{N}^B \mathrm{d}\rho\right)}$$

Our task:

- choose a maximal subset of the fields $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho : \Sigma \to \mathbb{R},\, \rho^i = (\partial_\rho)^i\}$

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \geq 0}$$

Solution 1°: using the inverse mean curvature flow (IMCF)

- choose a maximal subset of the fields $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho:\Sigma\to\mathbb{R}\,, \rho^i=(\partial_\rho)^i\}$

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \geq 0}$$

- what is if we keep (Σ, h_{ij}) but drop $\rho:\Sigma\to\mathbb{R}$ and the shift from $\rho^i=(\partial_\rho)^i$

The foliation and part of the flow is to be determined dynamically

- the **inverse mean curvature flow**

$$\boxed{\rho^i_{\{IMCF\}} = \left(\widehat{K}^l{}_l\right)^{-1}\widehat{n}^i + \widehat{N}^i_{\{IMCF\}}}$$

- as for the corresponding foliation $\widehat{N}\widehat{K}^l{}_l \equiv 1$ hold: if this flow existed globally the Geroch mass would be non-decreasing w.r.t it
- one can relax these condition by using a generalized IMCF

$$\boxed{\rho^i = \mathscr{L}_\rho(\log[\mathscr{A}_\rho])\,\rho^i_{\{IMCF\}}}$$

- **(!)** global existence and regularity is a serious issue

Solution 2°: using globally well-behaving foliation ...

- choose a maximal subset of the fields $\{\widehat N, \widehat N^A, \widehat\gamma_{AB}\,;\, \rho:\Sigma\to\mathbb{R}\,, \rho^i=(\partial_\rho)^i\}$

$$\boxed{\widehat N \widehat K^l{}_l = \overline{\widehat N \widehat K^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{{}^{(3)}R + \mathring{\widehat K}_{kl}\mathring{\widehat K}^{kl} + 2\,\widehat N^{-2}\,\widehat\gamma^{kl}\,(\widehat D_k \widehat N)(\widehat D_l \widehat N) \geq 0}$$

- what is if we drop the three-metric h_{ij} BUT keep the globally well-defined foliation $\rho:\Sigma\to\mathbb{R}$, with a flow $\rho^i=(\partial_\rho)^i$ and with induced metric $\widehat\gamma_{AB}$ on the leaves

Using prescribed foliation, flow, induced metric: $h_{ij} \leftrightarrow \widehat N,\ \widehat N^A,\ \widehat\gamma_{AB}$

- $\boxed{\rho^i = \widehat N\,\widehat n^i + \widehat N^i}$ however counterintuitive it is we may always construct shift $\widehat N^i$ with desirable properties:

$$\boxed{\widehat N \widehat K^l{}_l = \tfrac{1}{2}\,\widehat\gamma^{ij}\mathscr{L}_\rho\widehat\gamma_{ij} - \widehat D_i \widehat N^i}$$

- or equivalently, as $\widehat N \widehat K^l{}_l = \overline{\widehat N \widehat K^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ is supposed to hold,

$$\boxed{\widehat D_A \widehat N^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat\gamma_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho] \qquad (**)}$$

Solution 2°: using prescribed foliation, flow and $\widehat{\gamma}_{AB}$

Solving $\widehat{D}_A \widehat{N}^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat{\gamma}_{ij})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ (**) on $\mathscr{S}_\rho$

- on topological two-spheres using then the Hodge decomposition of the shift $\boxed{\widehat{N}^A = \widehat{D}^A\,\chi + \widehat{\epsilon}^{AB}\,\widehat{D}_B\,\eta}$, χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\boxed{\widehat{D}^A \widehat{D}_A \chi = \mathscr{L}_\rho \log[\,\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

- solubility in terms of spherical harmonics presumes that some standard polar coordinates (ϑ, φ), given on the unit sphere $\mathbb{S}^2$, are transfered to $\mathscr{S}$
- by Lie dragging polar coordinates (ϑ, φ) along the prescribed flow $\rho^i = (\partial_\rho)^i$ (**) can be solved in a synchronized way on each of the leaves throughout Σ

Not done yet (!) ${}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \geq 0$

- in clearing up the picture let us have a glance again of the key equation

$$\boxed{{}^{(3)}R = \widehat{R} - \left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\right\} \qquad (*)}$$

A parabolic equation for $\widehat{N}$:

- as noticed first by Bartnik (1993), while applying quasi-spherical foliations (*) can be viewed as a parabolic equation for $\widehat{N}$
- remarkably, (*) **can always be seen to be a parabolic eqn** for $\widehat{N}$ **IF** ${}^{(3)}R$, $\widehat{\gamma}_{AB}$ and $\widehat{N}^A$ can be treated as prescribed fields
- introducing $\boxed{\overset{\star}{K}_{ij} = \widehat{N}\widehat{K}_{ij}}$ and $\boxed{\overset{\star}{K} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_i\widehat{N}^i}$ to **eliminate hidden occurrence** of the lapse in (*) we get

$$\boxed{\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\hat{D}_l\widehat{N})\,] = \widehat{N}^2(\widehat{D}^l\widehat{N}_l\widehat{N}) + \mathcal{A}\,\widehat{N} - \tfrac{1}{2}\,(\,\widehat{R} - {}^{(3)}R\,)\,\widehat{N}^3}$$

where $\boxed{\mathcal{A} = \partial_\rho\overset{\star}{K} + \tfrac{1}{2}\,[\overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}]}$ with $\boxed{\overset{\star}{K} = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho\log[\mathscr{A}_\rho] > 0}$

- it is standard to obtain **existence of unique solutions to this (Bernoulli type) uniformly parabolic PDE** in a sufficiently small one-sided neighborhood of $\mathscr{S}$ in Σ

Global existence of unique solutions:

- our main concern is **global existence (!)**
- it should not come as a surprise that **an analogous parabolic equation** came up **in deriving the evolutionary form** of the Hamiltonian constraints in [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]
- if, e.g., $\boxed{{}^{(3)}R + \hat{\mathring{K}}_{kl}\hat{\mathring{K}}^{kl} = 0}$ **global unique solutions exist** to

$$\boxed{\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\hat{D}_l\widehat{N})\,] = \widehat{N}^2(\widehat{D}^l\widehat{N}_l\widehat{N}) + (\partial_\rho \overset{\star}{K} + \tfrac{3}{4}\,\overset{\star}{K}{}^2)\,\widehat{N} - \tfrac{1}{2}\,\widehat{R}\,\widehat{N}^3}$$

- for **any smooth positive initial data** ${}_0\widehat{N}$ on some $\mathscr{S}_{\rho_0}$ a unique **positive bounded** solution $\widehat{N}$ **exists for all** $\rho \geq \rho_0$
- if $\Sigma \approx \mathbb{R}^3$ and the freely specifiable data $\widehat{\gamma}_{AB}$ is chosen such that suitable integral terms approximate their "asymptotically flat forms" then in the $\rho \to \infty$ limit $\widehat{N} \to 1$ can also be guaranteed

Summary:

a **simple construction** of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that

1. the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
2. the foliation gets to be **quasi-convex** w.r.t. the constructed three-metric h_{ij} : $\widehat{K}^l{}_l = \widehat{N}^{-1} \mathscr{L}_\rho \log[\mathscr{A}_\rho]$
3. **ultimate aim** is to construct initial data sets with these properties
4. the topology of Σ **could be** either $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{R} \times \mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^2$
5. the construction **applies to wide range of geometrized theories of gravity**
    - no use of Einstein's equations or any other field equation on the metric of the ambient space had been applied anywhere in our construction
    - as only the Riemannian character of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.