Ultrarelativistic boost of a black hole in the magnetic universe of Levi-Civita–Bertotti–Robinson (LCBR)*

Marcello Ortaggio

Institute of Mathematics of the Czech Academy of Sciences

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Contents

The AS *pp*-wave

Aichelburg-Sexl'70,'71:

1. take the Schwarzschild metric
2. *boost* it to the speed v
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   use the identity $\lim_{\epsilon \to 0} \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) = \delta(x) \int_{-\infty}^{\infty} f(t) \mathrm{d}t$

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$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 - 4\sqrt{2}p\ln(x^2+y^2)\delta(u)\mathrm{d}u^2,$$

with $T_{uu} \propto p\delta(x)\delta(y)\delta(u)$.

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- all curvature invariants vanish
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  [Amati-Klimčík'89, Horowitz-Steif'90]
- used as "incoming state":
  high-speed black hole encounters [Curtis'78, D'Eath'78,...]
  semi-classical approaches to Planckian scattering and BH production "in the lab" ['t Hooft'87, Eardley-Giddings'02,...]

Other constructions of impulsive waves:

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- limits of sandwich waves.

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Generalizations to various backgrounds . . . (and dimensions)

Backgrounds	cut-and-paste	sandwich waves	AS boost
Minkowski	Penrose'68,'72 Dray-'t Hooft'85 (*) Sfetsos'95	Brinkmann'25 Peres'60, Kundt'61 Bondi *et al.*'62,. . . Bonnor'69	Aichelburg-Sexl'70 Loustó-Sanchez'89,'90 Balasin-Nachbagauer'95 Barrabés-Hogan'01,. . .
(A)dS	Hotta-Tanaka'93 Podolský-Griffiths'99 Horowitz-Itzhaki'99	García D.-Plebański'81 Ozsváth *et al.*'83 Siklos'85	Hotta-Tanaka'93
Bonnor-Melvin	(cf. *)	Garfinkle-Melvin'92	M.O.'04
LCBR, Nariai	(cf. *)	García D.-Alvarez C.'84 Khlebnikov'86 Lewandowski'92 Podolský-M.O.'03	??

The Alekseev-Garcia solution

$$\mathrm{d}s^2 = -e^{2\psi}\cosh^2\frac{z}{b}\mathrm{d}t^2 + e^{2\gamma}(\mathrm{d}z^2 + \mathrm{d}\rho^2) + e^{-2\psi}b^2\sin^2\frac{\rho}{b}\mathrm{d}\phi^2,$$

coordinates (t, z, ρ, ϕ), $\rho \in [0, \pi b]$, $\phi \in [0, 2\pi]$; 3 parameters m, b, l

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$$e^{2\psi} = \frac{\left(R_+ + R_- - 2m\cos\frac{\rho}{b}\right)^2}{(R_+ + R_-)^2 - 4m^2},$$

$$e^{2\gamma} = \frac{\left(R_+ + R_- - 2m\cos\frac{\rho}{b}\right)^2}{4R_+R_-}\left[\frac{R_+ - b\sinh\frac{z}{b} + (l+m)\cos\frac{\rho}{b}}{R_- - b\sinh\frac{z}{b} + (l-m)\cos\frac{\rho}{b}}\right]^2,$$

$$R_\pm^2 = \left(l \pm m - b\sinh\frac{z}{b}\cos\frac{\rho}{b}\right)^2 + b^2\cosh^2\frac{z}{b}\sin^2\frac{\rho}{b},$$

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Horizon at: $\rho = 0$, $l - m \leq b\sinh\frac{z}{b} \leq l + m$.

$\boxed{m = 0}$:

LCBR solution $\mathrm{AdS}_2 \times S_2$ [Levi-Civita'17, Bertotti'59, Robinson'59]

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$\boxed{b \to +\infty}$: (exterior) Schwarzschild in Weyl coordinates

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$$R_\pm^2 = (\pm m - \tilde{z})^2 + \rho^2 \qquad (\tilde{z} = z - l).$$

Ultrarelativistic limit of the AG solution

Idea: decompose

$$\mathrm{d}s^2_{AG} = (\mathrm{d}s^2_{AdS_2} + \mathrm{d}s^2_{S_2}) + \Delta \qquad (\Delta = \ldots)$$

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(Note $U = 0 \Rightarrow Z_2 = \pm b$.)

Eventually:

$$\mathrm{d}s^2 = \frac{2\mathrm{d}u\mathrm{d}v + 2H\delta(u)\mathrm{d}u^2}{\Omega^2} + \mathrm{d}\rho^2 + b^2\sin^2\frac{\rho}{b}\mathrm{d}\phi^2,$$

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$$\Omega = 1 - \frac{1}{2}b^{-2}uv, \qquad H = \sqrt{2}p\frac{b^2}{b^2 + l^2}\ln\frac{1 + \cos\frac{\rho}{b}}{1 - \cos\frac{\rho}{b}},$$

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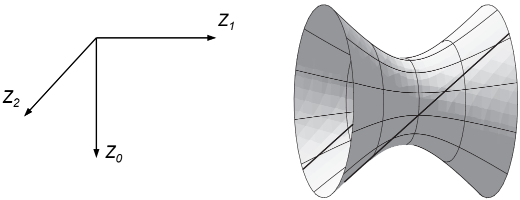
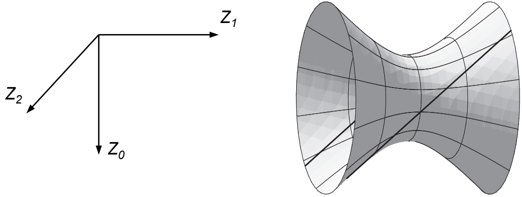
LCBR coordinates given by:

$$
\begin{aligned}
b\cosh\frac{z}{b}\cos\frac{t}{b} &= \frac{u-v}{\sqrt{2}\Omega}, \qquad b\sinh\frac{z}{b} = \frac{u+v}{\sqrt{2}\Omega}, \\
b\cosh\frac{z}{b}\sin\frac{t}{b} &= (\pm)b\left(\frac{2}{\Omega}-1\right).
\end{aligned}
$$


Z_1
Z_2
Z_0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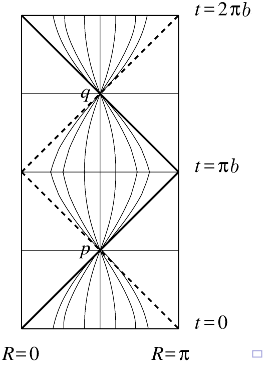
$t=2\pi b$
q
$t=\pi b$
p
$t=0$
$R=0$
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Kundt form: under $\tilde{v} = \frac{v}{\Omega}$, $\zeta = \sqrt{2} b e^{i\phi} \tan \frac{\rho}{2b}$

$$ds^2 = 2\mathrm{d}u\mathrm{d}\tilde{v} + \left[-b^{-2}\tilde{v}^2 + 2\tilde{H}(u,\zeta,\bar{\zeta})\right]\mathrm{d}u^2 + \frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{\left(1+\frac{1}{2}b^{-2}\zeta\bar{\zeta}\right)^2},$$

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