Complex and real type $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces

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Goals of the presentation

- Classification of the complex type [N] $\otimes$ [N] spaces
- New examples of the Lorentzian slices of the complex type [N] $\otimes$ [N] metrics

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Type [N] in Lorentzian geometry

Why type [N] is so interesting in General Theory of Relativity?

- Peeling Theorem and possible relation between type [N] and gravitational waves

$$C_{abcd} = \frac{[\mathrm{N}]}{\lambda} + \frac{[\mathrm{III}]}{\lambda^2} + \frac{[\mathrm{II}]}{\lambda^3} + \frac{[\mathrm{I}]}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right)$$

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Different classes of Lorentzian type [N] metrics

There are 3 vacuum classes of Lorentzian type [N] metrics

- Kundt class (nontwisting, nonexpanding, pp-waves as a special subclass)
- Robinson - Trautman class (nontwisting, expanding)
- Twisting class. The only known explicit solution is *Hauser solution*[1] which is equipped with two symmetries (one Killing vector[2], one homothetic vector[3])

[1] I. Hauser, *Type - N gravitational field with twist*, Phys. Rev. Lett. **33**, 1112 (1974)

[2] K_a is a Killing vector if $\nabla_{(a}K_{b)} = 0$.

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Why complex approach?

Advantages:

- In complex spaces which SD Weyl spinor is algebraically degenerate, Einstein vacuum field equations have been reduced to the single *hyperheavenly equation*
- The results are valid in 4-dimensional spaces with the neutral signature metric $(++--)$

Disadvantage:

- No general techniques which lead to the Lorentzian slices

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Type [N] in Lorentzian geometry

Penrose theorem says:

$$C_{ABCD} = m_{(A}n_B r_C s_{D)}, \; C_{\dot{A}\dot{B}\dot{C}\dot{D}} = m_{(\dot{A}}n_{\dot{B}} r_{\dot{C}} s_{\dot{D})}$$

where m_A, n_A, r_A and s_A are *undotted Penrose spinors*, $m_{\dot{A}}$, $n_{\dot{A}}$, $r_{\dot{A}}$ and $s_{\dot{A}}$ are *dotted Penrose spinors*.

Type [N] is characterized by the condition

$$C_{ABCD} = m_A m_B m_C m_D$$

In Lorentzian geometry: $m_{\dot{A}} = \overline{m_A}$, so

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The first necessary condition for the existence of the Lorentzian slices

In complex geometry there is no relation between undotted and dotted Penrose spinors, so there exist spaces of the "mixed" types, like $[\mathrm{N}] \otimes [\mathrm{D}]$.

The first necessary condition for existing of the Lorentzian slices

If a complex space admits real Lorentzian slice then SD and ASD parts of the Weyl spinor are of the same Petrov-Penrose type[a].

[a] K. Rózga, *Real slices of complex space-time in general relativity*, Rep. Math. Phys. **11**, 197 (1977)

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Congruence of the SD null strings

Let $\mathcal{D}$ be a 2-dimensional SD distribution: $\mathcal{D} = \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{A}} b^{\dot{A}} \neq 0$. It is integrable in the Frobenius sense, if

$$m^A m^B \nabla_{A\dot{M}} m_B = 0. \tag{1}$$

Equations (1) are called *SD null string equations*.

Integral manifolds of the distribution $\mathcal{D}$ are 2-dimensional, holomorphic, totally null and geodesic surfaces, called *null strings*. Their family constitutes *the congruence of the SD null strings*.

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Congruence of the SD null strings

SD null string equations can be rewritten in the form

$$m^B \nabla_{A\dot{M}} m_B = m_A M_{\dot{M}}.$$

Spinor field $M_{\dot{M}}$ is called *expansion of the congruence*.

- $M_{\dot{M}} = 0$ – *nonexpanding congruence*
- $M_{\dot{M}} \neq 0$ – *expanding congruence*

Nonexpanding congruence = distribution $\mathcal{D}$ is parallely propagated:

$$\nabla_V X \in \mathcal{D} \text{ for any vector field } V \text{ and any vector field } X \in \mathcal{D}$$

Spaces which admit nonexpanding congruence of SD null strings are called *Walker spaces*.

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The second necessary condition for the existence of the Lorentzian slices

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If a complex space equipped with the congruences of SD and ASD null strings admits real Lorentzian slice then both congruences are expanding or both are nonexpanding.

The first criterion of the classification: properties of the congruences of the null strings:

- $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$
- $[\mathrm{N}]^n \otimes [\mathrm{N}]^n$
- $[\mathrm{N}]^n \otimes [\mathrm{N}]^e$ (this type does not admit Lorentzian slices)

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Intersection of SD and ASD congruences of the null strings

Intersection of these congruences constitutes the congruence of the complex, null geodesics. It is given by the vector field $K_a \sim m_A m_{\dot{B}}$. Define *complex expansion* θ and *complex twist* ϱ by the formulas

$$\theta := \frac{1}{2}\nabla^a K_a \quad \sim \quad m_A M^A + m_{\dot{A}} M^{\dot{A}}$$

$$\varrho^2 := \frac{1}{2}\nabla_{[a} K_{b]} \, \nabla^a K^b \quad \sim \quad m_A M^A - m_{\dot{A}} M^{\dot{A}}$$

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- $[-+]$: $\theta = 0, \varrho \neq 0$ (this case cannot appear in Einstein spaces)
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Type $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces

There are 6 subtypes of the complex type $[\mathrm{N}] \otimes [\mathrm{N}]$ Einstein spaces:

- $\{[\mathrm{N}]^n \otimes [\mathrm{N}]^n, [--]\}$ (complex equivalent of pp-waves)[4]
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Hyperheavenly spaces - definition

Definition

$Hyperheavenly\ space$ ($\mathcal{HH}$-space) is a 4-dimensional complex analytic differential manifold equipped with a holomorphic metric ds^2 which satisfies the vacuum Einstein equations and such that the self-dual part of the Weyl tensor is algebraically degenerate.

Hyperheavenly spaces - the metric

The metric of the Einstein type $[\mathrm{N}] \otimes [\mathrm{any}]$ spaces can be brought to the form[8]

$$\begin{aligned} ds^2 &= 2\phi^{-2}\{(d\eta dw - d\phi dt) - \phi\, W_{\eta\eta}\, dt^2 \\ &\qquad +(2W_\eta - 2\phi\, W_{\eta\phi})\, dwdt + (2W_\phi - \phi\, W_{\phi\phi})\, dw^2\} \end{aligned}$$

where (ϕ, η, w, t) are local coordinates called *Plebański - Robinson - Finley coordinates*, function $W = W(\phi, \eta, w, t)$ is *the key function*, which satisfies *the hyperheavenly equation*

$$\phi W_{\eta\eta} W_{\phi\phi} - \phi W_{\eta\phi} W_{\eta\phi} + 2W_\eta W_{\eta\phi} - 2W_\phi W_{\eta\eta} + (W_{w\eta} - W_{t\phi}) = \phi\gamma$$

$\gamma = \gamma(w, t)$ is an arbitrary function such that $\gamma_t \neq 0$.

[8] J.F. Plebański, I. Robinson, *Left - degenerate vacuum metrics*, Phys. Rev. Lett. **37**, 493 (1976)

Type $\{[\mathrm{N}]^n \otimes [\mathrm{N}]^n, [--]\}$

The metric reads

$$ds^2 = 2(d\zeta d\widetilde{\zeta} - dvdu + (f(u,\zeta) + \widetilde{f}(u,\widetilde{\zeta}))du^2) \tag{2}$$

where $f = f(u,\zeta)$ and $\widetilde{f}(u,\widetilde{\zeta})$ are arbitrary holomorphic functions such that $f_{\zeta\zeta} \neq 0$, $\widetilde{f}_{\widetilde{\zeta}\widetilde{\zeta}} \neq 0$. The metric admits null Killing vector $K = \dfrac{\partial}{\partial v}$.

- $(u, v, \zeta, \widetilde{\zeta})$ – real, the metric (2) has neutral signature
- (u, v) – real, $(\zeta, \widetilde{\zeta} = \bar{\zeta})$ – complex with $\widetilde{f} = \bar{f}$, the metric (2) has Lorentzian signature and it is *pp-wave metric*

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The metric reads

$$ds^2 = 2\phi^{-2}\left\{\frac{1}{\tau}(d\eta dw - d\phi dt) - 2\phi^2 dt^2 + \left(2\eta^2 - (g_x + f_t)\frac{\phi}{2\tau}\right) dw^2\right\}$$

where

$$g = g(x, w), \ g_{xxx} \neq 0, \ x := t - \frac{1}{2\tau\phi}, \ f = f(w, t), \ f_{ttt} \neq 0$$

There exist complex transformation of the variables which brings the metric to the form

$$\begin{aligned} ds^2 \quad &= \quad 2d\zeta d\widetilde{\zeta} - 2du\left(dv - \frac{2v}{\zeta + \widetilde{\zeta}} d(\zeta + \widetilde{\zeta})\right) \\ &\quad +2\left(\frac{v^2}{(\zeta + \widetilde{\zeta})^2} - (\zeta + \widetilde{\zeta})(H(u, \zeta) + \widetilde{H}(u, \widetilde{\zeta}))\right) du^2 \end{aligned} \tag{3}$$

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Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [--]\}$

Similarity between *pp-wave metric* and *Kundt class*

- Both spaces are equipped with nontwisting and nonexpanding congruence of null geodesics

Differences between *pp-wave metric* and *Kundt class*

- Pp-waves are equipped with null Killing vector, while Kundt class does not admit such symmetry
- Complex pp-waves are equipped with congruences of SD and ASD the null strings, both nonexpanding: $\{[\mathrm{N}]^n \otimes [\mathrm{N}]^n, [--]\}$ while complex Kundt class is equipped with congruences of SD and ASD the null strings, both expanding: $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [--]\}$

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Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [+-]\}$, the first sub-type

The metric of the first sub-type can be brought to the form

$$ds^2 = 2\phi^{-2}\left\{\frac{1}{\tau}e^{-2\tau f}dwdx - \frac{1}{\tau}d\phi dt - \phi(f_t + g_x)dt^2 + 2e^{-2\tau f}g\,dwdt\right\} \quad (4)$$

where $g = g(x,t)$ such that $g_{xxx} \neq 0$ and $f = f(w,t)$ such that $(\tau f_w^2 + f_{ww})_t \neq 0$.
There exist complex transformation of the variables which brings the metric (4) to the form

$$ds^2 = -2drdu + 2r\,\partial_u \ln(H\widetilde{H})\,du^2 + \frac{2r^2}{H^2\widetilde{H}^2}d\zeta d\widetilde{\zeta},\ H = H(u,\zeta),\ \widetilde{H} = \widetilde{H}(u,\widetilde{\zeta}) \quad (5)$$

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[9] H. Stephani at al., "Exact Solutions...", Theorem 28.1 specialized for $m = 0$ and $\Delta \ln P = 0$, $\Delta := P^2\partial_\zeta\partial_{\bar{\zeta}}$.

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Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [+-]\}$, the second sub-type

The metric of the second sub-type can be brought to the form

$$\begin{aligned} ds^2 \quad &= \quad 2\phi^{-2}\{\tau^{-1}(d\eta dw - d\phi dt) - \phi(F_{\eta\eta} + 2\phi)dt^2 \\ &\qquad +2F_\eta \, dwdt + 2(\eta^2 + m)dw^2\} \end{aligned} \tag{6}$$

where $F = F(\eta, w, t)$ and $m = m(w, t)$ satisfy the following equation

$$2(\eta^2 + m)F_{\eta\eta} - 4\eta F_\eta - \frac{1}{\tau}F_{\eta w} + \frac{1}{\tau}m_t = 0, \;\; F_{\eta\eta\eta\eta} \neq 0, \;\; m_t \neq 0$$

General solution is not know.

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [+-]\}$, the second sub-type

The metric (6) can be brought to the form

$$ds^2 = -2drdu - \left(r\,\partial_u \ln(KP^{-2}) + \varepsilon\right)du^2 + \frac{2r^2}{P^2}\frac{K}{\varepsilon}\,d\zeta d\widetilde{\zeta} \tag{7}$$

where $\varepsilon = \pm 1$, functions $P = P(u, \zeta, \widetilde{\zeta})$ and $K = K(u, \zeta)$ satisfy the equation

$$2P^2(\ln P)_{\zeta\widetilde{\zeta}} = K \tag{8}$$

- $(u, v, \zeta, \widetilde{\zeta})$ – real, the metric (7) has neutral signature
- (u, v) – real, $(\zeta, \widetilde{\zeta} = \bar{\zeta})$ – complex, the metric (7) has Lorentzian signature and it belongs to the *Robinson - Trautman class*[10]

Moreover, Lorentzian slice implies:

- Function K becomes the function of only one variable u and admissible gauge freedom allows to bring it to the constant value, $K = \varepsilon = \pm 1$
- Equation (8) reduces to the equation $2P^2(\ln P)_{\zeta\bar{\zeta}} = \pm 1$

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[10] H. Stephani at al., "Exact Solutions...", Theorem 28.1 specialized for $m = 0$ and $\Delta \ln P = \pm 1$, $\Delta := P^2 \partial_\zeta \partial_{\bar{\zeta}}$.

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[10] H. Stephani at al., "Exact Solutions...", Theorem 28.1 specialized for $m = 0$ and $\Delta \ln P = \pm 1$, $\Delta := P^2 \partial_\zeta \partial_{\bar{\zeta}}$.

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$ without any symmetries:

- The hyperheavenly equation splits into the overdetermined system of three equations for two functions of three variables

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- A little simpler system but still overdetermined

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Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$, two symmetries

In local coordinates (v, ϕ, w, t) the metric can be brought to the form

$$
\begin{aligned}
ds^2 = \frac{2}{\phi^2}\Bigg\{ & \frac{1}{\tau}\left(t^{1-2\chi_0} - \phi\frac{dh}{dv}\right) dvdw - \frac{1}{\tau} h\, d\phi dw - \frac{1}{\tau}\, d\phi dt \\
& - \left(\phi t^{-1}\left(\frac{dT}{dv} - \frac{1-2\chi_0}{2\tau}\right) - \phi^2 t^{2\chi_0-2}\left(h\frac{d^2T}{dv^2} - \frac{d^2Z}{dv^2}\right)\right) dt^2 \\
& + 2\left(t^{-2\chi_0}T - \phi h t^{-1}\left(\frac{dT}{dv} - \frac{1-2\chi_0}{2\tau}\right) + \frac{1}{2}\phi^2 t^{2\chi_0-2}\left(h^2\frac{d^2T}{dv^2} - \frac{dP}{dv}\right)\right) dwdt \\
& + \left(2t^{-2\chi_0}Z + \phi t^{-1}\left(P - 2h\frac{dZ}{dv}\right) + \phi^2 t^{2\chi_0-2} h\left(h\frac{d^2Z}{dv^2} - \frac{dP}{dv}\right)\right) dw^2 \Bigg\}
\end{aligned}
$$

where $P = P(v)$, $Z = Z(v)$, $T = T(v)$ and

$$
h := \frac{Z'''}{T'''} \equiv \frac{P''}{Z'''}, \quad ' := \partial_v, \quad T''' \neq 0, \quad h \neq \text{const}
$$

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$, two symmetries

Where

$$Z(v) := \frac{1}{Q'}, \; T(v) := \frac{1}{2\tau}\frac{Q}{Q'}$$
$$P(v) := \tau Q^{\chi_0 - 1} Q'^{-\frac{1}{2}} \int Q^{-\chi_0} Q'^{\frac{3}{2}} \left(\frac{\gamma_0}{\tau^2} - 2Q'^{-2}\{Q, v\} \right) dv$$

where $Q = Q(v)$ and

$$\{Q, v\} := \frac{Q'''}{Q'} - \frac{3}{2}\frac{Q''^2}{Q'^2}$$

is the *Schwarzian derivative* of the function Q

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$, two symmetries

Function $Q(v)$ satisfies the fifth-order ODE

$$\left(\frac{2Z'''Q'^{-1}}{\Delta T'''}\{Q,v\}\right)' - \frac{1}{2\Delta}[QQ'^{-2}Q'' + 2(\chi_0 - 2)]\left[\frac{T'''}{\Delta}\left(2Q'^{-2}\{Q,v\} - \frac{\gamma_0}{\tau^2}\right) + \frac{Z'''^2}{T'''}\right] = 0$$

where

$$Z''' = -Q'(Q'^{-2}\{Q,v\})'$$
$$T''' = -\frac{1}{2\tau}\frac{Q'}{Q}\left(\frac{Q^2}{Q'^2}\{Q,v\}\right)'$$
$$\Delta = -\frac{1}{2\tau^2}\frac{Q^2}{Q'^2}\{Q,v\} + \frac{(\chi_0 - 2)(\chi_0 - 1)}{\tau^2}$$

Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$, two symmetries

Disadvantages of our approach:

- No new solutions have been found so far (most promising case is $\chi_0 = 2$)
- Hauser solution has not been reconstructed so far
- No transformation which reduce the order of the final differential equation has been found so far

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Advantages of our approach

- Final equation is ODE and it can be written in the form

$$Q''''' = G(Q, Q', Q'', Q''', Q'''')$$

  with G being the rational function. It always has a solution for arbitrary initial values. It works in complex case, real Lorentzian case and real neutral case.
- We formulated the theorem which is complex counterpart of the theorem formulated by W.D. Halford (1979) and C.D. Collinson (1969, 1980)

Theorem

For any vacuum $\mathcal{HH}$-spaces of the type $[N] \otimes [II,D,III,N]$ with twisting congruence of null geodesics arising as intersection of SD null strings with ASD null strings there exist at most two homothetic Killing vectors. They must be noncommuting.

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Type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^n, [--]\}$

In neutral signature spaces:

$$\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [--]\} \longrightarrow \{[\mathrm{N}]^e \otimes [\mathrm{N}]^n, [--]\} \longrightarrow \{[\mathrm{N}]^n \otimes [\mathrm{N}]^n, [--]\}$$

In this case the metric can be brought to the form

$$\begin{aligned} ds^2 &= 2\phi^{-2}\left\{\tau^{-1}(d\eta dw - d\phi dt) - 2B_0\phi^2 dwdt \right. \\ &\quad \left. + (2g - \phi g_\phi - f\phi + 2B_0\eta\phi)dw^2\right\} \end{aligned} \tag{9}$$

where

$$f = f(w,t), \; f_{tt} \neq 0, \; g = g(\phi, w), \; g_{\phi\phi\phi} \neq 0, \; B_0 = \{1, 0\}$$

If $B_0 = 0$ then the metric (9) admits null Killing vector $K = \partial_\eta$.

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The metric reads

$$\begin{aligned} ds^2 \;=\; & 2\phi^{-2}\{\tau^{-1}(d\eta dw - d\phi dt) - \phi^2 T_{xx}\, dt^2 \\ & +(2\phi\eta T_{xx} - 2\phi^2 T_x)\, dwdt \\ & +(2\phi\eta T_x - \eta^2 T_{xx} - \tau^{-1} C\phi)\, dw^2\} \end{aligned}$$

where $x := \eta/\phi$, $C = C(w,t)$ is an arbitrary function such that $C_{tt} \neq 0$, function $T = T(x,w,t)$ satisfies the equation

$$C\, T_{xx} + T_{xw} - 3T_t + xT_{xt} = 0$$

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One symmetry: $K_1 = \partial_w - 2\chi_0 \left(\phi\partial_\phi + \eta\partial_\eta\right)$

The metric reads

$$\begin{aligned} ds^2 \quad = \quad & 2\phi^{-2}\{\tau^{-1}(d\eta dw - d\phi dt) - \phi^2 e^{2\chi_0 w} H_{xx}\, dt^2 \\ & +2e^{2\chi_0 w}\phi^2(xH_{xx} - H_x)\, dwdt \\ & +(2B_0\eta - \tau^{-1}C\phi - e^{2\chi_0 w}\eta\phi(xH_{xx} - 2H_x))\, dw^2\} \end{aligned}$$

where $x := \eta/\phi$, $C = C(t)$ is an arbitrary function such that $C_{tt} \neq 0$, function $H = H(x,t)$ satisfies the equation

$$(-2\tau B_0 x + C)H_{xx} + (2\tau B_0 + 2\chi_0)H_x + xH_{xt} - 3H_t = 0$$

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& \left[2\phi^2 t^{2(\chi_0-1)}\left(h\frac{dU}{dh} - U\right) - 2\Omega_0\phi h t^{-1}\right]dwdt \\
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for $\chi_0 = 1$ this equation has the form very similar to the equation describing Hauser solution.

[11] A.C., M. Przanowski, *On twisting type $[\mathrm{N}] \otimes [\mathrm{N}]$ Ricci flat complex spacetimes with two homothetic symmetries*, Journal of Mathematical Physics 59, 042504 (2018)

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Concluding Remarks

- Complex twisting class with one symmetry and without any symmetries: can it be reduced to the single equation?
- Some tricks used in analysis of the complex types $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces have been successfully used to find first explicit example of para-Hermite space of the type $\{[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^n, [++,++]\}$.

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