q-de Sitter and k-Poincaré symmetries in Chern-Simons (2+1)D gravity with cosmological constant

Giacomo Rosati

INFN Cagliari

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Wojanów Palace

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Outlook and motivations

(Lukierski et al.,Majid et al.'90,Ballesteros et al.2000s-) (Amelino-Camelia et al.2000s-)

- q-de Sitter $\xrightarrow{\Lambda\to 0}$ κ-Poincaré $\longrightarrow$ Quantum Gravity / Planck-scale DSR phenomenology: e.g. gamma ray bursts, astrophysical neutrinos. Interplay between comsological expansion $\Lambda \sim 10^{-28}\text{sec}^{-2}$ and $\kappa \sim M_{pl} \sim 10^{19}\text{GeV}$

- Do they arise from more fundamental approaches to quantum gravity?
  - Low energy limit of 4D QG, some hints: (Amelino-Camelia+Starodubtsev+Smolin2004, Smolin2006 Girelli+Livine2010)
  - 3D gravity: theory is topological → more manageable
    Some evidence for κ-symmetries (Freidel+Kowalski-Glikman+Smolin2004, Freidel+Livine2006)

    However: more systematic analysis (Moesburger+Schroers2009) 3D gravity as Chern-Simons, particle as "puncture", quantized phase space encoded in r-matrix

    negative result! *The r-matrix used in the literature for κ-Poincaré is not compatible with the scalar product defining the Chern-Simons action*

- New results (G.R.2017) give positive answer
  - Different r-matrix constructively derived from first principles.
  - Use an approach relying on the "quantum duality principle" (Drinfeld1987, Semenov-Tian-Shansky1992) between Hopf algebras and (dual) Poisson Lie groups proposed in (Ballesteros+Musso2013) to derive the quantized particle phase space and the deformed relativistic symmetries

Outline

1. (2+1)D de Sitter spacetime symmetries
   - Splitting the algebra in two SU(2)
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3. Chern-Simons (2+1)D Λ-gravity with particle
   - Action
   - Particle phase space
   - r-matrix
4. The quantum duality principle: from the dual Lie bialgebras to the Hopf algebra
   - Dual Lie bialgebra
   - Dual Lie group
   - Poisson structure
   - q-de Sitter
   - $\Lambda \to 0$ contraction to κ-Poincaré
   - Bicrossproduct basis
   - q-de Sitter non-commutative spacetime

(2+1)D de Sitter spacetime symmetries

(2+1)dS

$$[E, P_a] = \Lambda N_a, \qquad [P_1, P_2] = \Lambda M,$$
$$[N_a, E] = -P_a, \quad [N_a, P_b] = -\delta_{ab} E, \quad [N_1, N_2] = -M,$$
$$[M, N_a] = \epsilon_{ab} N_b, \quad [M, P_a] = \epsilon_{ab} P_b, \quad [M, E] = 0,$$

(2+1)dS ≅ SO(3,1)

$$E = -\sqrt{\Lambda} K_3, \quad P_a = -\sqrt{\Lambda} \epsilon_{ab} J_b, \quad M = J_3, \quad N_a = -K_a,$$
$$\left[J_i, J_j\right] = \epsilon_{ijk} J_k, \quad \left[J_i, K_j\right] = \epsilon_{ijk} K_k, \quad \left[K_i, K_j\right] = -\epsilon_{ijk} J_k,$$

In 4D gravity homogeneous isotropic universe with positive cosmological constant driven expansion: $H = \dot{a}/a \sim \sqrt{\Lambda} \sim 10^{-19} \mathrm{sec}^{-1}$. Maximally symmetric: 10 spacetime symmetry generators (Bacry+Levy-Leblond1968, Cacciatori+Gorini+Kamenshchik2016)

(2+1)D de Sitter spacetime symmetries

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(2+1)dS ≅ SO(3,1)

$$E = -\sqrt{\Lambda} K_3, \quad P_a = -\sqrt{\Lambda}\epsilon_{ab} J_b, \quad M = J_3, \quad N_a = -K_a,$$
$$\left[J_i, J_j\right] = \epsilon_{ijk} J_k, \quad \left[J_i, K_j\right] = \epsilon_{ijk} K_k, \quad \left[K_i, K_j\right] = -\epsilon_{ijk} J_k,$$

matrix representation $\mathfrak{so}(3,1)$

$$\rho(J_i) = \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}, \quad \rho(K_i) = \mathcal{M}_{0i}, \quad (\mathcal{M}_{AB})_{KL} = \eta_{AK}\delta_{BL} - \eta_{BK}\delta_{AL}$$
$$[\mathcal{M}_{AB}, \mathcal{M}_{CD}] = \{\eta_{AD}\mathcal{M}_{BC} + \eta_{BC}\mathcal{M}_{AD} - \eta_{AC}\mathcal{M}_{BD} - \eta_{BD}\mathcal{M}_{AC}\}$$
$$g = \exp\left(\tfrac{1}{2}\alpha^{AB}\mathcal{M}_{AB}\right) = \exp\left(j_i\rho(J_i) + k_i\rho(K_i)\right), \quad g \in \mathrm{SO}(3,1)$$

$\mathfrak{so}(3,1)_{\mathbb{C}} \approx \mathfrak{su}(2) \oplus_{\mathbb{C}} \mathfrak{su}(2)$

$$L_i = \frac{1}{2}(J_i + iK_i), \quad R_i = \frac{1}{2}(J_i - iK_i)$$
$$\left[L_i, L_j\right] = \epsilon_{ijk} L_k, \quad \left[R_i, R_j\right] = \epsilon_{ijk} R_k, \quad \left[L_i, R_j\right] = 0$$
$$g = g_l g_r = \exp\{l_i\rho(L_i)\}\exp\{r_i\rho(R_i)\}$$
$$l_i = j_i - ik_i, \qquad r_i = j_i + ik_i, \qquad l_i = r_i^*$$

Cartan-Weyl basis

$$H^L = iL_3, \quad X_\pm^L = i(L_1 \pm iL_2)$$
$$H^R = iR_3, \quad X_\pm^R = i(R_1 \pm iR_2)$$
$$\left[H^I, H^J\right] = 0, \quad \left[H^I, X_\pm^J\right] = \pm\delta_{IJ} X_\pm^J$$
$$\left[X_+^I, X_-^J\right] = 2\delta_{IJ} H^J, \quad I, J = L, R$$

3D gravity basis

$$\mathcal{J}_0 = -J_3, \quad \mathcal{J}_a = K_a,$$
$$\mathcal{P}_0 = \sqrt{\Lambda} K_3, \quad \mathcal{P}_a = \sqrt{\Lambda} J_a,$$
$$\left[\mathcal{J}_\mu, \mathcal{J}_\nu\right] = \epsilon_{\mu\nu\rho}\mathcal{J}^\rho, \quad \left[\mathcal{J}_\mu, \mathcal{P}_\nu\right] = \epsilon_{\mu\nu\rho}\mathcal{P}^\rho,$$
$$\left[\mathcal{P}_\mu, \mathcal{P}_\nu\right] = -\Lambda\epsilon_{\mu\nu\rho}\mathcal{J}^\rho,$$

$$C_1 = \mathcal{P}_\mu\mathcal{P}^\mu - \Lambda\mathcal{J}_\mu\mathcal{J}^\mu, \qquad C_2 = \mathcal{P}_\mu\mathcal{J}^\mu$$

reality conditions

$$J_i^* = -J_i, \qquad K_i^* = K_i$$

$$\mathcal{J}_0^* = -\mathcal{J}_0, \quad \mathcal{J}_a^* = \mathcal{J}_a,$$
$$\mathcal{P}_0^* = \mathcal{P}_0, \quad \mathcal{P}_a^* = -\mathcal{P}_a.$$

Chern-Simons (2+1)D Λ-gravity with particle

(Witten1988, Alekseev+Malkin1994, Meusburger+Schroers2000*)

- Chern-Simons action with de Sitter as gauge group G
- manifold $\mathcal{M} = \Sigma \times \mathbb{R}$: 2D Riemannian surface (space) - segment of the real line (time)
- particles as punctures (conical defects) on Σ.

$$S = \kappa \int_{\mathbb{R}} dt \int_{\Sigma} \left\langle \partial_t A_\Sigma \wedge A_\Sigma - \phi(\xi_0) g^{-1} \partial_t g + A_0 \left(\kappa F_\Sigma - \phi(\xi) \delta^2(\vec{x} - \vec{x}') dx^1 \wedge dx^2 \right) \right\rangle_B$$

coupling constant

$$\kappa = \frac{1}{4\pi G} \sim M_{pl}$$

Cartan connection

$$A = \omega_\mu \mathcal{J}^\mu + e_\mu \mathcal{P}^\mu$$

$$A = A_0 dt + A_\Sigma$$

$$F_\Sigma = dA_\Sigma + A_\Sigma \wedge A_\Sigma$$

inner product

$$C_2 = \mathcal{P}_\mu \mathcal{J}^\mu$$

$$\Downarrow$$

$$B(\mathcal{J}_\mu, \mathcal{P}_\nu) = \eta_{\mu\nu},$$

$$B(\mathcal{J}_\mu, \mathcal{J}_\nu) = B(\mathcal{P}_\mu, \mathcal{P}_\nu) = 0$$

coadjoint orbit

$$\left.\begin{array}{l} \xi_0 = m\tilde{\mathcal{P}}_0 + s\tilde{\mathcal{J}}_0, \\ \xi = \mathrm{Ad}^* \xi_0 = p^\mu \tilde{\mathcal{P}}_\mu + j^\mu \tilde{\mathcal{J}}_\mu \end{array}\right. \in \mathfrak{g}^*$$

$$\langle A \wedge A \rangle_B = B(e_i, e_j) \left(A^i \wedge A^j \right)$$

duality

$$\{\mathcal{P}_\mu, \mathcal{J}_\mu\} = \{e_j\} \in \mathfrak{g} \qquad \langle \tilde{e}_i, e_j \rangle = \delta_{ij}$$

$$\{\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{J}}_\mu\} = \{\tilde{e}_j\} \in \mathfrak{g}^*$$

$$\langle X, \mathrm{Ad}_g^* \tilde{Y} \rangle = \langle g^{-1} X g, \tilde{Y} \rangle$$

$$\phi(\tilde{\mathcal{J}}_\mu) = P_\mu, \quad \phi(\tilde{\mathcal{P}}_\mu) = J_\mu$$

phase space

- The particle's momenta are the parameters of the dual Poisson-Lie group (G^*, λ) associated to the coadjoint orbit $\begin{cases} \xi = \mathrm{Ad}^*_g \xi_0 = p^\mu \tilde{\mathcal{P}}_\mu + j^\mu \tilde{\mathcal{J}}_\mu \\ \xi_0 = m\tilde{\mathcal{P}}_0 + s\tilde{\mathcal{J}}_0 \end{cases} \in \mathfrak{g}^*$

- The infinitesimal counterpart of (G^*, λ) is the Lie bialgebra $(\mathfrak{g}^*, \delta^*)$ dual to the coboundary Lie bialgebra $(\mathfrak{g}, \delta)$ with co-commutator $\delta(X) = \sum([X, r_{z(1)}]\otimes 1 + 1\otimes[X, r_{z(2)}])$

- the r-matrix $r_z = z(r_+ + r_-)$ must be compatible with the inner product defining the CS action

$$C_2 = \mathcal{P}_\mu \mathcal{J}^\mu \Rightarrow \begin{array}{c} B(\mathcal{J}_\mu, \mathcal{P}_\nu) = \eta_{\mu\nu}, \\ B(\mathcal{J}_\mu, \mathcal{J}_\nu) = B(\mathcal{P}_\mu, \mathcal{P}_\nu) = 0 \end{array} \Rightarrow r_+ = \mathcal{J}_\mu \otimes \mathcal{P}^\mu + \mathcal{P}_\mu \otimes \mathcal{J}^\mu$$

- z can be interpreted as a quantum deformation parameter to be determined

- The quantization of (G^*, λ) provides the Hopf algebra $(U_z(\mathfrak{g}), \Delta_z)$ (quantum duality principle) of the relativistic symmetry generators corresponding to the particle's momenta

$$(G^*, \lambda, \Delta^*_G) \longrightarrow (\hat{G}^*, \Delta) \cong (U_z(\mathfrak{g}), \Delta_z)$$

r-matrix

$$r_z = z(r_+ + r_-)$$

symmetric part

$$r_+ = \mathcal{J}_\mu \otimes \mathcal{P}^\mu + \mathcal{P}_\mu \otimes \mathcal{J}^\mu$$

antisymmetric part

classical Yang-Baxter equation (CYBE)

$$[\![r, r]\!] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

$$r_{ij} = \sum 1\otimes\cdots\otimes r^{(1)}\otimes 1\otimes\cdots\otimes r^{(2)}\otimes\cdots\otimes 1$$

$\rightarrow$ $(\mathfrak{g}, \delta)$ is coboundary and quasitriangular

$$r_- = i m_\rho \epsilon^{\rho\mu\nu}\left(\mathcal{J}_\mu \otimes \mathcal{P}_\nu + \mathcal{P}_\mu \otimes \mathcal{J}_\nu\right)$$

$$m^\mu m_\mu = 1, \quad m_\mu = (1, 0, 0)$$

deformation parameter

1. $r_- \in \mathbb{R} \Rightarrow \delta \in \mathbb{R} \Rightarrow \mathfrak{g}^* \in \mathbb{R}$

   $\Rightarrow z$ purely imaginary

2. $[z] = [\mathcal{P}_0]^{-1} \Rightarrow [z] \propto \kappa^{-1}$

   $\Rightarrow r_z$ is "timelike"

$$\mathcal{J}_0^* = -\mathcal{J}_0, \quad \mathcal{J}_a^* = \mathcal{J}_a,$$

$$\mathcal{P}_0^* = \mathcal{P}_0, \quad \mathcal{P}_a^* = -\mathcal{P}_a$$

reality conditions

$$r^{(*\otimes *)} = \tau(r)$$

$$\tau(a\otimes b) = (b\otimes a)$$

$\mathcal{R}$ is real $\Leftrightarrow \mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$

"quantum inverse Killing form"

is self-adjoint

(MajidFoundations1995)

$$r_z = \frac{i}{\kappa}\left(\mathcal{J}_\mu \otimes \mathcal{P}^\mu + \mathcal{P}_\mu \otimes \mathcal{J}^\mu\right) - \frac{1}{\kappa}\epsilon^{ij}\left(\mathcal{J}_i \otimes \mathcal{P}_j + \mathcal{P}_i \otimes \mathcal{J}_j\right)$$

$$r_z = r_L + r_R, \qquad r_{L,R} = z_{L,R}\left(H^{L,R} \otimes H^{L,R} + X_+^{L,R} \otimes X_-^L\right), \qquad z_L = -z_R = 2\sqrt{\Lambda}/\kappa$$

$$U_q(\mathfrak{su}(2)) : \qquad q = \exp z, \qquad \mathcal{R}^{(*\otimes *)} = \tau(\mathcal{R})$$

$$q_L = q_R^{-1}$$

(Cianfrani+Kowalski-Glikman+Pranzetti+G.R.2016)

Dual Lie bialgebra

$$\boxed{r = z\,(H \otimes H + X_+ \otimes X_-)}$$

Aim: derive the Poisson-Lie structure λ on G^*,
i.e. the Poisson brackets between the parameters of G^*: $g^* \equiv \exp(p_i e^i) \quad \{e^i\} \in \mathfrak{g}^*$

dual Lie bialgebra

$$\mathfrak{g} : \left[e_i, e_j\right] = c_{ij}{}^k e_k, \quad \delta : \delta(e_i) = f^{ij}{}_k e_i \otimes e_j$$

$$\mathfrak{g}^* : \left[e^i, e^j\right] = f^{ij}{}_k e^k, \quad \delta^* : \delta^*\left(e^i\right) = c_{jk}{}^i e^j \otimes e^k$$

$$[H, X_\pm] = \pm X_\pm \quad [X_+, X_-] = 2H$$

$$\delta(H) = 0, \quad \delta(X_+) = zX_+ \wedge H, \quad \delta(X_-) = zX_- \wedge H$$

$$\left[\tilde{H}, \tilde{X}_\pm\right] = -z\tilde{X}_\pm, \quad \left[\tilde{X}_+, \tilde{X}_-\right] = 0$$

$$\delta^*\left(\tilde{H}\right) = 2\tilde{X}_+ \wedge \tilde{X}_-, \quad \delta^*\left(\tilde{X}_\pm\right) = \pm\tilde{H} \wedge \tilde{X}_\pm$$

$$a \wedge b = a \otimes b - b \otimes a$$

$$\cancel{\delta^*\left(e^i\right) = \tilde{r}_{A(1)} \otimes \left[e^i, \tilde{r}_{A(2)}\right] + \left[e^i, \tilde{r}_{A(1)}\right] \otimes \tilde{r}_{A(2)}} \Rightarrow \cancel{\{a, b\} = \frac{1}{2} r^{ij} \cdot ((X_i^R \wedge X_j^R - X_i^L \wedge X_j^L)(a \otimes b))}$$

Sklyanin bracket

Constructive procedure based on the quantum duality principle proposed by (Ballesteros+Musso2013)

Dual Lie group

adjoint representation of $\mathfrak{g}^*$: $\quad (\rho(\tilde{e}_i))^j_{\ k} = -f^{ij}_{\ \ k}$

$$\rho\left(\tilde{H}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix}, \quad \rho\left(\tilde{X}_+\right) = \begin{pmatrix} 0 & -z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho\left(\tilde{X}_-\right) = \begin{pmatrix} 0 & 0 & -z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Element of G^*

$$g^* = \exp\left(h\rho\left(\tilde{H}\right)\right)\exp\left(x_+\rho\left(\tilde{X}_+\right)\right)\exp\left(x_-\rho\left(\tilde{X}_-\right)\right) = \begin{pmatrix} 1 & -zx_+ & -zx_- \\ 0 & e^{zh} & 0 \\ 0 & 0 & e^{zh} \end{pmatrix}$$

Alternative parametrization

$$g^{*\prime} = \exp\left(h'\rho\left(\tilde{H}\right)/2\right)\exp\left(x'_+\rho\left(\tilde{X}_+\right)\right)\exp\left(x'_-\rho\left(\tilde{X}_-\right)\right)\exp\left(h'\rho\left(\tilde{H}\right)/2\right) = \begin{pmatrix} 1 & -ze^{zh'/2}x'_+ & -ze^{zh'/2}x'_- \\ 0 & e^{zh'} & 0 \\ 0 & 0 & e^{zh'} \end{pmatrix}$$

$$\tilde{h} = h, \qquad \tilde{x}_\pm = e^{-zh/2}x_\pm$$

Coproduct $\longleftrightarrow$ product in G^*

$$\begin{pmatrix} \Delta 1 & -z\Delta x_+ & -z\Delta x_- \\ 0 & \Delta e^{zh} & 0 \\ 0 & 0 & \Delta e^{zh} \end{pmatrix} = \begin{pmatrix} 1\otimes 1 & -zx_+\otimes 1 & -zx_-\otimes 1 \\ 0 & e^{zh}\otimes 1 & 0 \\ 0 & 0 & e^{zh}\otimes 1 \end{pmatrix}\begin{pmatrix} 1\otimes 1 & -z1\otimes x_+ & -z1\otimes x_- \\ 0 & 1\otimes e^{zh} & 0 \\ 0 & 0 & 1\otimes e^{zh} \end{pmatrix}$$

$$\Delta 1 = 1\otimes 1, \qquad \Delta h = h\otimes 1 + 1\otimes h, \qquad \Delta x_\pm = x_\pm \otimes e^{zh} + 1\otimes x_\pm$$

$$\Delta 1 = 1\otimes 1, \qquad \Delta h' = h'\otimes 1 + 1\otimes h', \qquad \Delta x'_\pm = x'_\pm \otimes e^{zh'/2} + e^{-zh'/2}\otimes x'_\pm$$

Poisson structure λ on G^*: $\quad g^* \equiv \exp(p_i e^i) \qquad \{e^i\} \in \mathfrak{g}^*$

Conditions on $\{\ ,\ \}_\lambda$

1. $\left\{\Delta(p_i), \Delta(p_j)\right\}_\lambda = \Delta\left(\left\{p_i, p_j\right\}_\lambda\right)$
2. $\{p_i, p_j\}_\lambda = c_{ij}{}^k + o(p^2)$

$$[e_i, e_j] = c_{ij}{}^k e_k, \qquad \{e_i\} \in \mathfrak{g}$$

solve analitically

$$\left\{x_i, x_j\right\}_\lambda = Q_{ij} = \sum_{k,l} \beta_{ijkl} F_k F_l$$

$$\{F_i\} := \left\{1, h, x_+, x_-, e^{zh}\right\}$$

$$\vdots$$

$$\{h, x_\pm\} = \pm x_\pm,$$

$$\{x_+, x_-\} = \frac{e^{2zh} - 1}{z} - z x_+ x_-$$

$$\{h', x'_\pm\} = \pm x'_\pm$$

$$\{x'_+, x'_-\} = 2\frac{\sinh(zh')}{z}$$

Quantize as a Hopf algebra:

$$(G^*, \lambda, \Delta^*_G) \longrightarrow (\hat{G}^*, \Delta) \cong (U_z(\mathfrak{g}), \Delta_z)$$

$$(h, x_\pm) \longrightarrow (\hat{H}, \hat{X}_\pm)$$

$$\{\ ,\ \} \longrightarrow [\ ,\ ]$$

q-de Sitter

crucial for the convergence of Δ when $\Lambda \to 0$

$U_q(\mathfrak{su}(2)) \oplus U_q^{-1}(\mathfrak{su}(2))$

(Cianfrani+Kowalski-Glikman+Pranzetti+G.R.2016)

(Celeghini+Giachetti+Sorace+Tarlini1991)

$$X_\pm \to \sqrt{\frac{z/2}{\sinh(z/2)}}X_\pm \qquad U_q(\mathfrak{su}(2))$$

$$[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{\sinh(zH)}{\sinh(z/2)}$$

$$\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta X_\pm = X_\pm \otimes e^{zH/2} + e^{-zH/2} \otimes X_\pm$$

$$z_L = -z_R = 2\sqrt{\Lambda}/\kappa$$

$$E = \sqrt{\Lambda}\left(H^L - H^R\right), \quad M = -i\left(H^L + H^R\right)$$

$$P_1 = \frac{\sqrt{\Lambda}}{2}\left(X_+^L - X_-^L + X_+^R - X_-^R\right)$$

$$P_2 = -\frac{i\sqrt{\Lambda}}{2}\left(X_+^L + X_-^L + X_+^R + X_-^R\right)$$

$$N_1 = \frac{1}{2}\left(X_+^L + X_-^L - X_+^R - X_-^R\right)$$

$$N_2 = -\frac{i}{2}\left(X_+^L - X_-^L - X_+^R + X_-^R\right)$$

$$[E, P_a] = \Lambda N_a, \quad [N_a, E] = -P_a$$

$$[P_1, P_2] = \Lambda \frac{\sin\left(\sqrt{\Lambda}M/\kappa\right)}{\sinh\left(\sqrt{\Lambda}/\kappa\right)} \cosh(E/\kappa)$$

$$[N_a, P_b] = -\delta_{ab}\sqrt{\Lambda}\frac{\sinh(E/\kappa)}{\sinh\left(\sqrt{\Lambda}/\kappa\right)} \cos\left(\sqrt{\Lambda}M/\kappa\right)$$

$$[N_1, N_2] = -\frac{\sin\left(\sqrt{\Lambda}M/\kappa\right)}{\sinh\left(\sqrt{\Lambda}/\kappa\right)} \cosh(E/\kappa)$$

$$[M, N_a] = \epsilon_a{}^b N_b, \quad [M, P_a] = \epsilon_a{}^b P_b, \quad [M, E] = 0$$

$$\Delta E = E \otimes 1 + 1 \otimes E, \quad \Delta M = M \otimes 1 + 1 \otimes M$$

$$\begin{aligned}\Delta P_a =& P_a \otimes e^{\frac{1}{2}E/\kappa} \cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) + e^{-\frac{1}{2}E/\kappa}\cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) \otimes P_a \\ &- \epsilon_{ab}\left(\sqrt{\Lambda}N_b \otimes e^{\frac{1}{2}E/\kappa}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) - \sqrt{\Lambda}e^{-\frac{1}{2}E/\kappa}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) \otimes N_b\right)\end{aligned}$$

$$\begin{aligned}\Delta N_a =& N_a \otimes e^{\frac{1}{2}E/\kappa} \cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) + e^{-\frac{1}{2}E/\kappa}\cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) \otimes N_a \\ &- \epsilon_{ab}\left(\frac{1}{\sqrt{\Lambda}}P_b \otimes e^{\frac{1}{2}E/\kappa}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) - \frac{1}{\sqrt{\Lambda}}e^{-\frac{1}{2}E/\kappa}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right) \otimes P_b\right)\end{aligned}$$

$\Lambda \to 0$ contraction to κ-Poincaré

$$[E, P_a] = 0, \qquad [P_1, P_2] = 0, \qquad [N_a, E] = -P_a$$

$$[N_a, P_b] = -\delta_{ab}\kappa \sinh(E/\kappa), \qquad [N_1, N_2] = -M \cosh(E/\kappa)$$

$$[M, N_a] = \epsilon_a{}^b N_b, \qquad [M, P_a] = \epsilon_a{}^b P_b, \qquad [M, E] = 0$$

$$\Delta E = E \otimes 1 + 1 \otimes E, \qquad \Delta M = M \otimes 1 + 1 \otimes M$$

$$\Delta P_a = P_a \otimes e^{\frac{1}{2}E/\kappa} + e^{-\frac{1}{2}E/\kappa} \otimes P_a$$

$$\Delta N_a = N_a \otimes e^{\frac{1}{2}E/\kappa} + e^{-\frac{1}{2}E/\kappa} \otimes N_a - \frac{1}{2\kappa}\epsilon_{ab}\left(P_b \otimes e^{\frac{1}{2}E/\kappa}M - e^{-\frac{1}{2}E/\kappa}M \otimes P_b\right)$$

Bicrossproduct basis

$$\tilde{E} = E, \qquad \tilde{M} = M,$$

$$\tilde{P}_a = e^{-\frac{1}{2\kappa}E}\left(\cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right)P_a - \epsilon_{ab}\sqrt{\Lambda}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right)N_b\right),$$

$$\tilde{N}_a = e^{-\frac{1}{2\kappa}E}\left(\cos\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right)N_a - \epsilon_{ab}\frac{1}{\sqrt{\Lambda}}\sin\left(\frac{\sqrt{\Lambda}}{2\kappa}M\right)P_b\right).$$

$$\boxed{\begin{aligned}
&\left[\tilde{E},\tilde{P}_a\right] = \Lambda\tilde{N}_a, \qquad \left[\tilde{N}_a,\tilde{E}\right] = -\tilde{P}_a,\\
&\left[\tilde{P}_1,\tilde{P}_2\right] = \frac{\Lambda}{2}\frac{\sin\left(2\sqrt{\Lambda}\tilde{M}/\kappa\right)}{\sinh\left(\sqrt{\Lambda}/\kappa\right)}, \qquad \left[\tilde{N}_1,\tilde{N}_2\right] = -\frac{1}{2}\frac{\sin\left(2\sqrt{\Lambda}\tilde{M}/\kappa\right)}{\sinh\left(\sqrt{\Lambda}/\kappa\right)},\\
&\left[\tilde{N}_a,\tilde{P}_b\right] = -\frac{\delta_{ab}\sqrt{\Lambda}}{\sinh(\frac{\sqrt{\Lambda}}{\kappa})}\left(\frac{1-e^{-2\tilde{E}/\kappa}}{2} - \sin^2\left(\frac{\sqrt{\Lambda}}{\kappa}\tilde{M}\right)\right)\\
&\qquad - \frac{\delta_{ab}}{2\kappa}\left(\tilde{\mathbf{P}}^2 - \Lambda\tilde{\mathbf{N}}^2\right) + \frac{1}{\kappa}\left(\tilde{P}_a\tilde{P}_b - \Lambda\tilde{N}_a\tilde{N}_b\right),\\
&\left[\tilde{M},\tilde{N}_a\right] = \epsilon_a{}^b\tilde{N}_b, \qquad \left[\tilde{M},\tilde{P}_a\right] = \epsilon_a{}^b\tilde{P}_b, \qquad \left[\tilde{M},\tilde{E}\right] = 0
\end{aligned}}$$

Contracts for $\Lambda \to 0$ to κ-Poincaré in bicrossproduct basis

q-de Sitter non-commutative spacetime

$$(\mathfrak{g},\delta) \longrightarrow G: \quad g = \exp\left(\tilde{h}H + \tilde{x}_+X_+ + \tilde{x}_-X_-\right)$$

$(\mathfrak{g},\delta)$ is coboundary

$\Downarrow$

Sklyanin bracket

$$\left\{\tilde{a},\tilde{b}\right\} = \frac{1}{2}r^{ij}\cdot((X_i^R \wedge X_j^R - X_i^L \wedge X^L)(\tilde{a}\otimes\tilde{b}))$$

$$X^R_{e_i}f(g) = \frac{d}{dt}\Big|_{t=0} f\left(e^{-te_i}g\right), \quad X^L_{e_i}f(g) = \frac{d}{dt}\Big|_{t=0} f\left(ge^{te_i}\right)$$

$$\left\{\tilde{h},\tilde{x}_\pm\right\} = -z\tilde{x}_\pm, \qquad \{\tilde{x}_+,\tilde{x}_-\} = 0$$

$$g_{dS} = \exp(tE + x^aP_a + \theta J + \xi^aN_a) = g_Lg_R$$

$$t = \frac{1}{2\sqrt{\Lambda}}\left(\tilde{h}^L - \tilde{h}^R\right), \qquad \theta = \frac{i}{2}\left(\tilde{h}^L + \tilde{h}^R\right)$$

$$x^1 = \frac{1}{2\sqrt{\Lambda}}\left(\tilde{x}_+^L - \tilde{x}_-^L + \tilde{x}_+^R - \tilde{x}_-^R\right)$$

$$x^2 = \frac{i}{2\sqrt{\Lambda}}\left(\tilde{x}_+^L + \tilde{x}_-^L + \tilde{x}_+^R + \tilde{x}_-^R\right)$$

$$\xi^1 = \frac{1}{2}\left(\tilde{x}_+^L + \tilde{x}_-^L - \tilde{x}_+^R - \tilde{x}_-^R\right)$$

$$\xi^2 = \frac{i}{2}\left(\tilde{x}_+^L - \tilde{x}_-^L - \tilde{x}_+^R + \tilde{x}_-^R\right)$$

$$[\hat{t},\hat{x}^a] = -\frac{1}{\kappa}\hat{x}^a, \qquad \left[\hat{t},\hat{\xi}^a\right] = -\frac{1}{\kappa}\hat{\xi}^a,$$

$$\left[\hat{\theta},\hat{x}^a\right] = -\frac{1}{\kappa}\epsilon^a{}_b\hat{\xi}^b, \qquad \left[\hat{\theta},\hat{\xi}^a\right] = -\frac{\Lambda}{\kappa}\epsilon^a{}_b\hat{x}^b$$

$$\left[\hat{\theta},\hat{t}\right] = \left[\hat{\xi}^a,\hat{\xi}^b\right] = \left[\hat{x}^a,\hat{x}^b\right] = \left[\hat{\xi}^a,\hat{x}^b\right] = 0.$$