Isolated horizons, the Petrov type D equation and the Near Horizon Geometry equation

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Denis Dobkowski-Ryłko, JL, Tomasz Pawłowski 2018, 2018
JL, Adam Szereszewski 2018
DDR, Wojciech Kamiński, JL, AS 2018, 2018

Extremal (degenerate) horizon equation

S - a 2d-manifold equipped with:

$g_{AB}dx^A dx^B$ - a metric tensor, $\omega_A dx^A$ - a 1-form

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$

K - the Gauss curvature

Λ - the cosmological constant

Hajicek 1974, Moncrief, Isenberg 1983, Ashtekar, Beetle, JL 2001, JL, Pawłowski 2002, Chruściel, Reall, Tod 2005, Jezierski 2009, Kunduri, J. Lucietti 2009, Chrusciel, Szybka, Tod 2017, Nurowski, Randall 2016

Extremal (degenerate) horizon equation

On a topological 2-sphere:

all the axisymmetric solutions known
Hajicek 1974, JL, Pawlowski 2002, Kunduri, J. Lucietti 2009,

unsolved existence problem for non-symmetric solutions
Chruściel, Reall, Tod 2005, Jezierski 2009, Chrusciel, Szybka, Tod 2017, Nurowski, Randall 2016, this talk

On a higher genus S: this talk, also *Chruściel, Reall, Tod 2005, C. Li, J. Lucietti 2013,*

Generalization to higher dimension and matter: *JL, Pawlowski 2005, Kunduri, J. Lucietti 2009,*

Near Horizon Geometry spacetime

Given a solution g_{AB}, ω_A on S

on: $S \times \mathbb{R} \times \mathbb{R}$

parametrized by: (x^A, u, v)

we define: $g_{\mu\nu}dx^\mu dx^\nu \quad :=$

$$g_{AB}dx^A dx^B \; - \; 2du\left(dv - 2v\omega - \frac{1}{2}v^2(\mathrm{div}\omega + 2\omega^2)du\right)$$

satisfies: $R_{\mu\nu} \; = \; 0$

Pawłowski, JL, Jezierski 2003, Kunduri, J. Lucietti 2009 - generalizations, *Bardeen, Horowitz 1999* - the explicit example obtained by the near the Kerr extremal horizon limit

The type D equation

S - a 2d-manifold $\quad g_{AB}dx^A dx^B = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$

$\omega_A dx^A$ - a 1-form

$d\omega =: \Omega \, d\text{Area}$ rotation pseudo scalar

$$\nabla^{(0,1)}T := \nabla_{\bar{z}}T \otimes d\bar{z}$$

$$(\nabla^{(0,1)})^2(K - \frac{\Lambda}{3} + i\Omega)^{-\frac{1}{3}} = 0$$

JL, Pawłowski 2001 $\Lambda = 0$, axi-symmetric solutions on topological sphere a local no-hair theorem

This equation knows the secret of the uniqueness of the Kerr black-hole

Local approach to black holes

Black holes in equilibrium:
non-expanding and shear-free null surfaces in 4d spacetime - horizons.

A. Ashtekar, C. Beetle, JL:

Mechanics of Rotating Isolated Horizons 2001

Geometry of Generic Isolated Horizons 2002

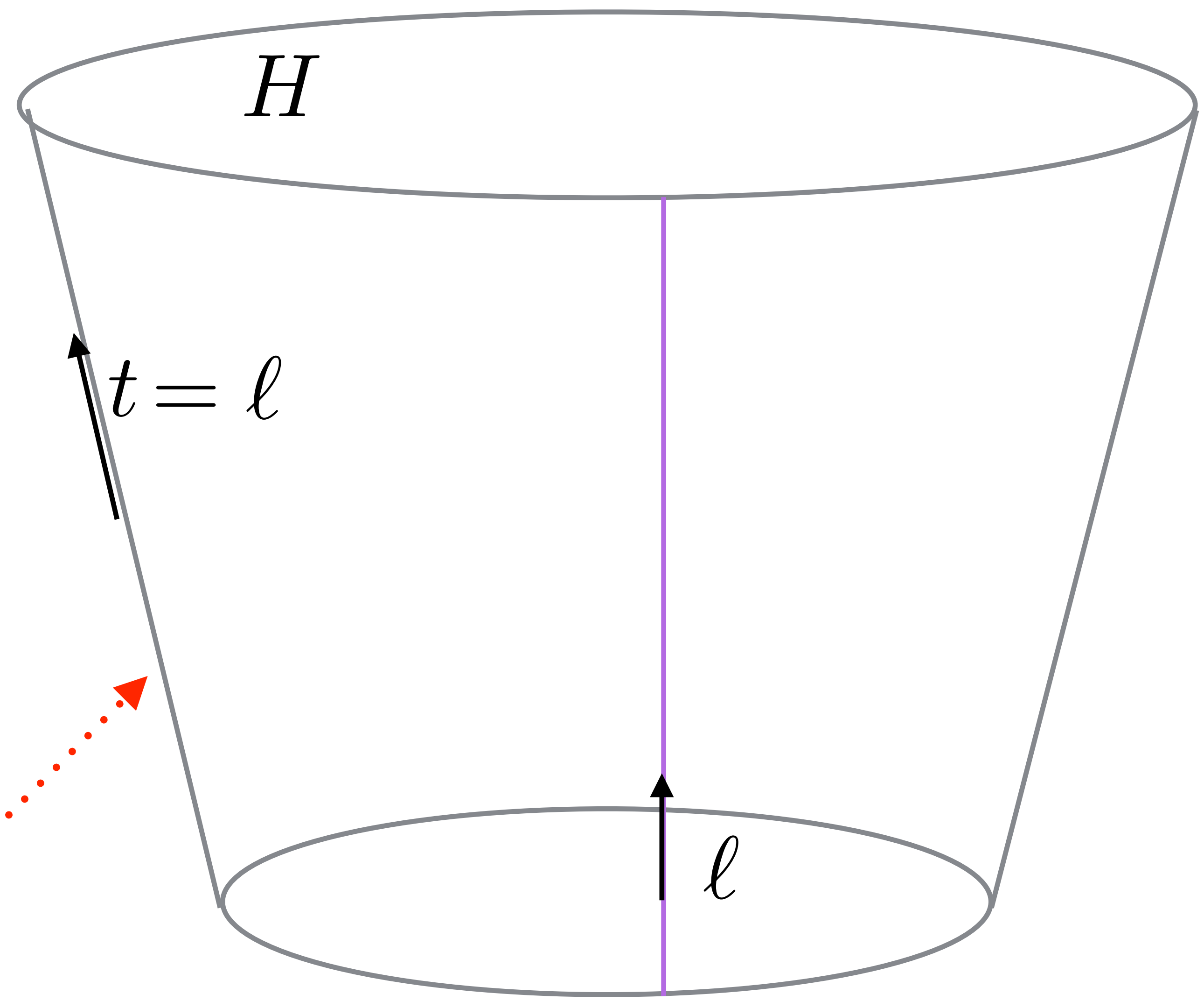
JL, Tomasz Pawłowski:
Geometric Characterizations of the Kerr Isolated Horizon 2002

Killing to the 2nd order horizon

3d null surface

in 4d spacetime $M \ , \ g_{\mu\nu}$

such that exists:



Such that

$$\left.\begin{aligned} \mathcal{L}_t g_{\mu\nu} &= 0 \\ [\mathcal{L}_t , \nabla_\mu] &= 0 \\ \mathcal{L}_t R_{\mu\nu\alpha\beta} &= 0 \end{aligned}\right\}$$

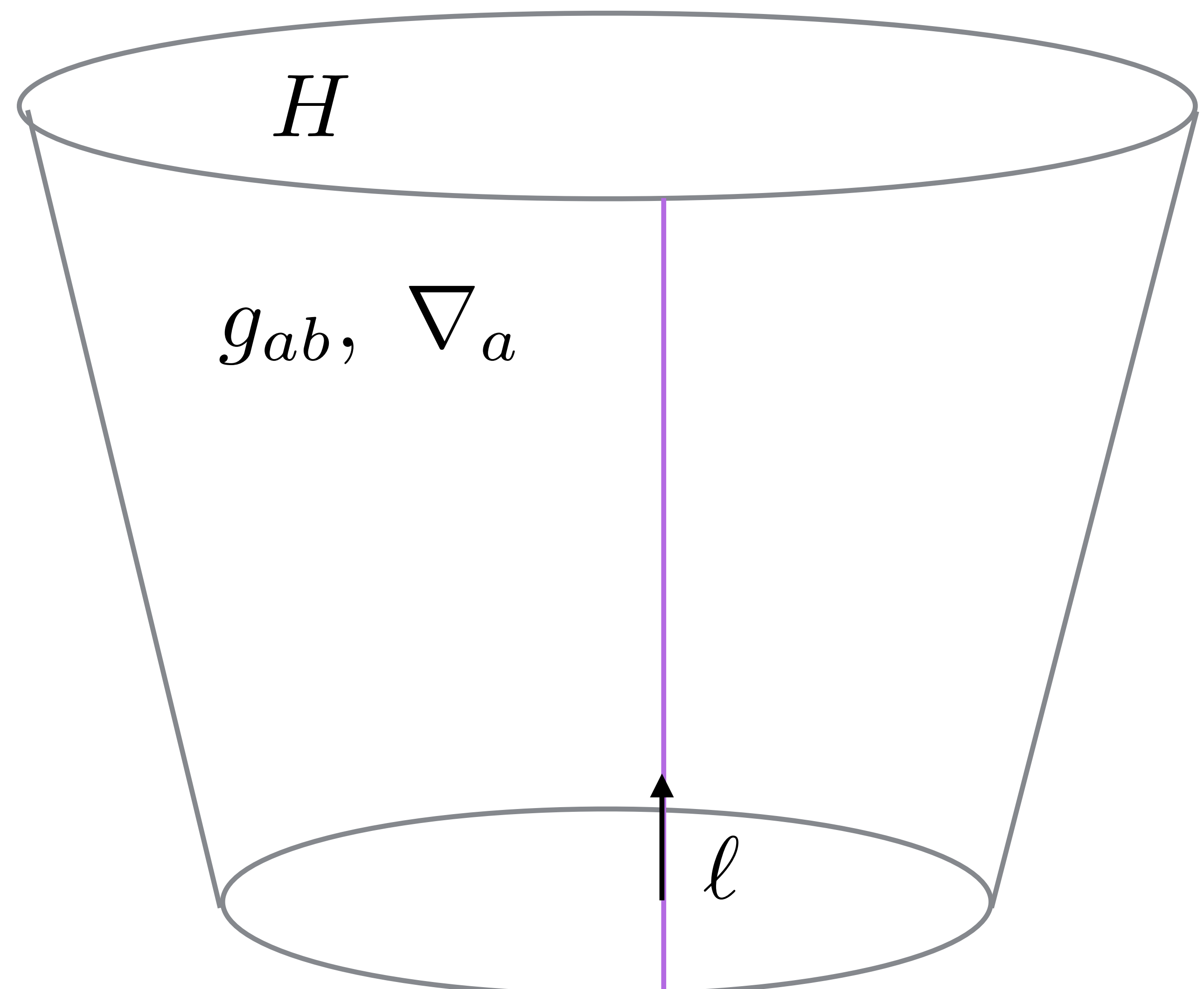
$$\ell^\mu \ell_\mu = \ 0$$

Assumption about $M \ , \ g_{\mu\nu}$ **:** $\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$

Resulting constraints on the horizon geometry

$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b$ rotation potential

$\omega_a^{(\ell)} \ell^a =: \kappa^{(\ell)}$ surface gravity



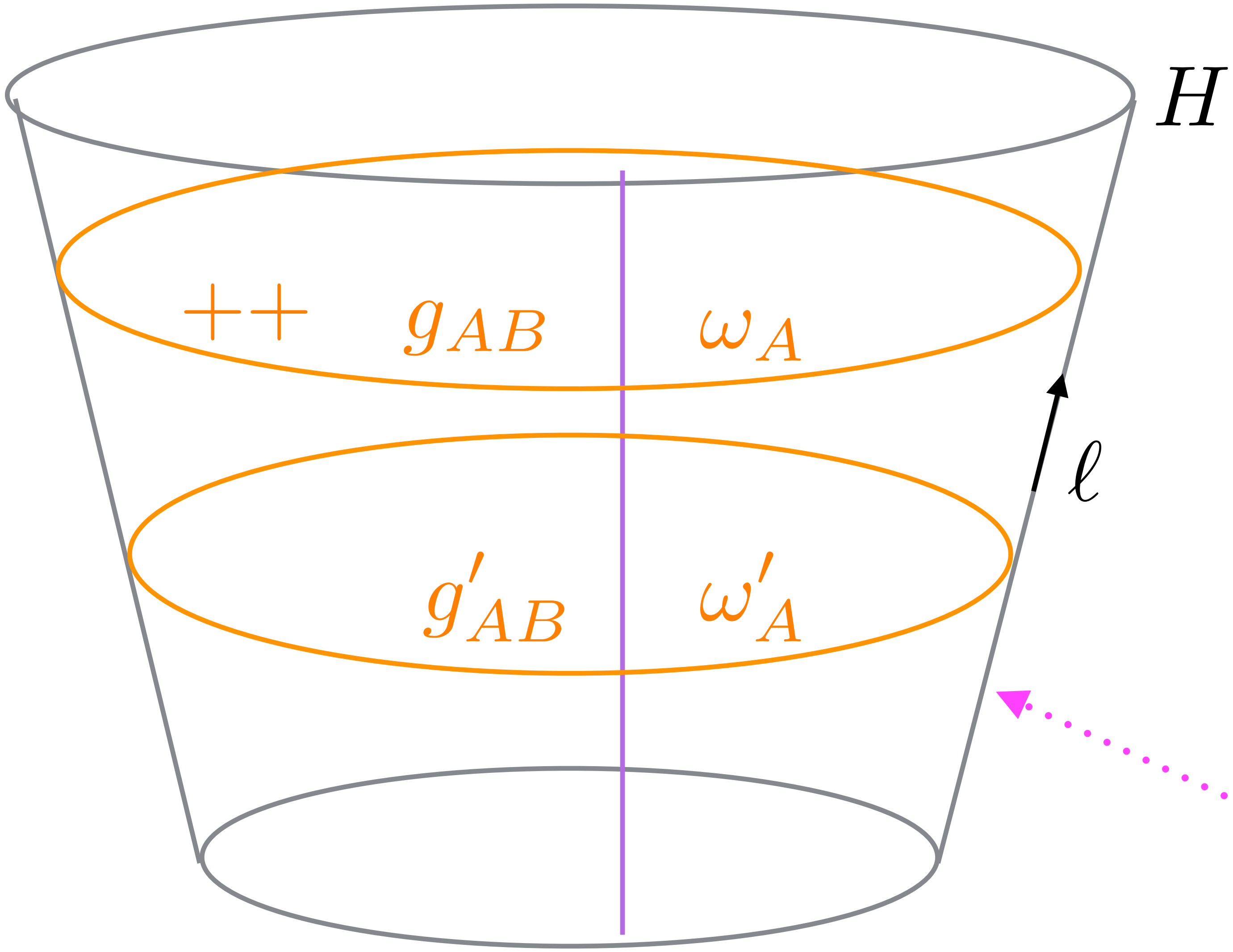
Assumption: $\kappa^{(\ell)} \neq 0$

i) $\kappa^{(\ell)} = \mathrm{const}$

The 0th low of BH thermodynamics

ii) $g_{ab}, \ \omega_a^{(\ell)}$ **determine all the** ∇_a

Free data on a 2d slice



$$g'_{AB} = g_{AB}$$
$$\omega'_A = \omega_A + f_{,A}$$

g_{AB}, ω_A **determines** $g_{ab}, \omega_a^{(\ell)}$ **determines** $g_{\mu\nu}, \nabla_\mu, R_{\mu\nu\alpha\beta}$

Technically important assumptions: $\ell \neq 0$ everywhere and the existence of a global section (see Dobkowski-Rylko's talk)

Horizon cross section geometry

2d-surface S

Endowed with (g_{AB}, ω_A) modulo $\omega'_A = \omega_A + f_{,A}$

∇_A - the corresponding derivative $\nabla_A g_{BC} = 0$ $\nabla_{[A}\nabla_{B]} f = 0$

Scalar invariants:

Gaussian curvature: K

Rotation scalar: $d\omega =: \Omega \; d\mathrm{Area}$ combined:

$$\Psi := -\frac{1}{2}(K + i\Omega)$$

Possible Petrov types

The spacetime Weyl tensor at H is determined by the data

$$(S, g_{AB}, \omega_A)$$

Theorem:

The possible Petrov types of *H* are:

~~I~~, II, D, ~~III~~, ~~N~~, O

wherein:

$$\Psi + \frac{\Lambda}{6} = 0 \quad \Leftrightarrow \quad \textbf{O} \quad \Leftrightarrow \quad K = \frac{\Lambda}{3} \qquad d\omega = 0$$

$\Psi + \frac{\Lambda}{6} \neq 0 \quad \Rightarrow$ **generically II, unless…**

The Petrov type D equation

We use a null 2-frame

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\mathrm{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

Theorem 1:

At *H* the spacetime Weyl tensor is of the Petrov type D iff the following two conditions are satisfied by the invariants of S:

$$\Psi + \frac{\Lambda}{6} \neq 0$$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = 0$$

Remark: a continues function $\left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}}$ exists locally on S

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\text{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}} = 0 \qquad \Psi + \frac{\Lambda}{6} \neq 0$$

Symmetries:

$$\phi : S \to S \qquad g' = \phi^* g,\ \omega' = \phi^* \omega$$

$$g' = g,\ \omega' = \omega + df$$

$$m'_A = \bar{m}_A,\ \omega'_A = \omega_A$$

The non-continuity of $(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}$**:** use a covering

$(S, g_{AB}, \omega_A) \to (\tilde{S}, \tilde{g}_{AB}, \tilde{\omega}_A)$ s.t. $(\tilde{\Psi} + \frac{\Lambda}{6})^{-\frac{1}{3}}$ continues

Relation with the Near Horizon Geometry Equation

Theorem 2:

Suppose (g_{AB}, ω_A) **satisfy the NHG equation, namely**

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$

and $\Psi + \frac{1}{6}\Lambda \neq 0$

Then they also satisfy the Petrov type D equation:

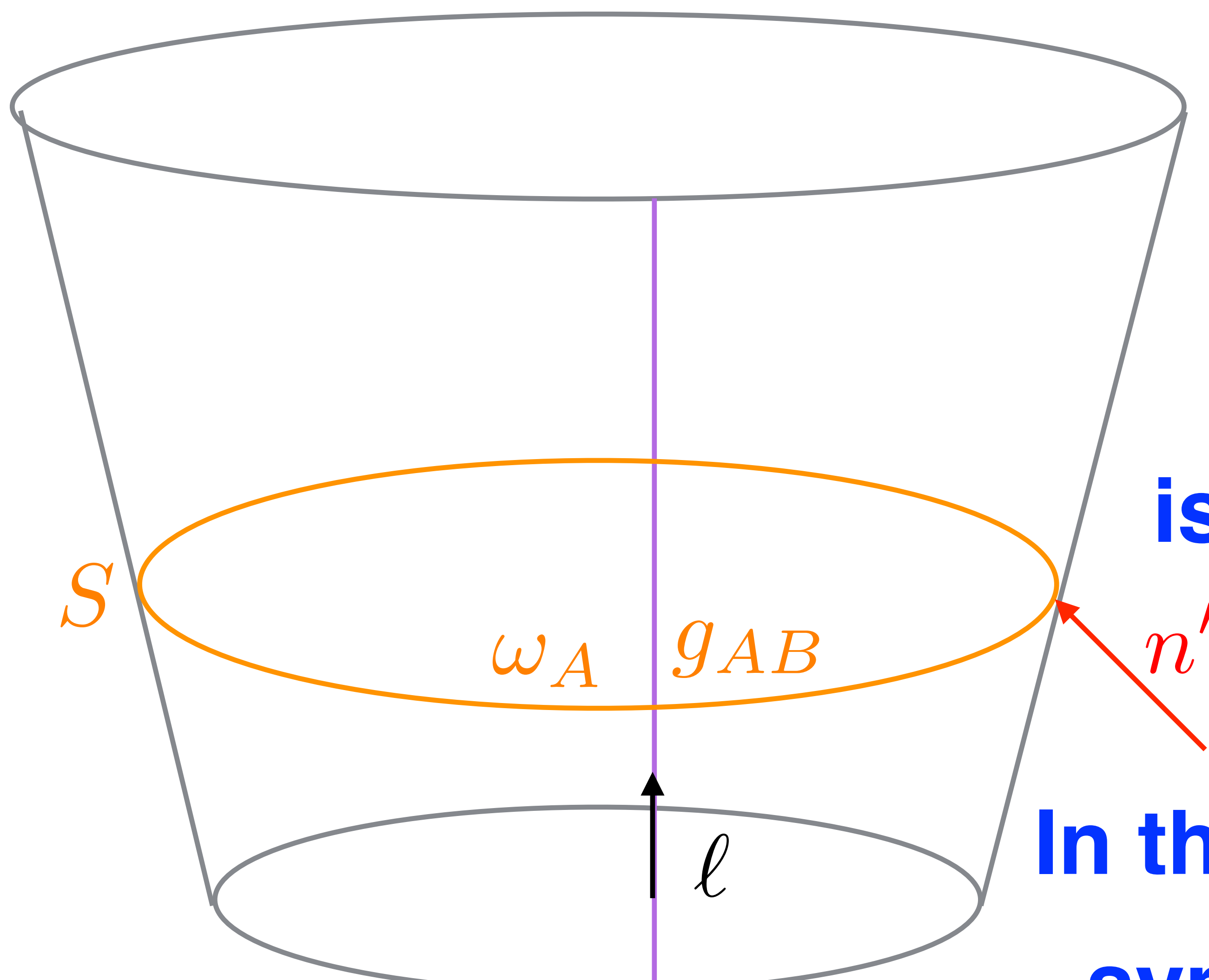
$$\bar{m}^A\bar{m}^B\nabla_A\nabla_B(\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Non-twisting
of the second principal null direction of the Weyl tensor

Theorem 3:

Suppose (g_{AB}, ω_A) **satisfy the NHG equation, namely**

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$



Then the null vector n' **orthogonal to the corresponding slice** S **is a double principal direction of the spacetime Weyl tensor at *H*.**

In that case there exists another symmetry *t'* that is extremal

the converse true for rotating

The holomorphic form of the type D equation

$$g_{AB}dx^A dx^B = \frac{2}{P^2}dz d\bar{z} \qquad\qquad m^A\partial_A = P\partial_z$$

$$z' = f(z)$$

$$X = X^z\partial_z + X^{\bar{z}}\partial_{\bar{z}} =: X^{(1,0)} + X^{(0,1)}$$

The Petrov type D equation reads:

$$\partial_{\bar{z}}(P^2\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0$$

$$\Leftrightarrow$$

$$X := g^{AB}\partial_B(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}\partial_A$$

$$X^{(1,0)}$$

is a holomorphic vector

The Petrov type D equation on S of genus > 0

$$g_{AB}dx^A dx^B = \frac{2}{P^2}dz d\bar{z} \qquad m^A\partial_A = P\partial_z$$

The Petrov type D equation: $$\partial_{\bar{z}}\left(P^2\partial_{\bar{z}}\left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}}\right) = 0$$

$$\Rightarrow \quad \partial_{\bar{z}}\left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}} = \frac{F(z)}{P^2}$$

$$\Rightarrow \quad F(z)\partial_z$$

is a globally defined holomorphic vector field

$$\Rightarrow \quad \begin{array}{ll} F(z) = \mathrm{const} & \text{if genus} = 1 \\ F(z) = 0 & \text{if genus} > 1 \end{array}$$

The Petrov type D equation on S of genus > 0

Theorem 4.
Suppose S is a compact 2-surface of genus >0.
The only solutions to the Petrov type D equation with a cosmological constant Λ are (g, ω) such that

$$d\omega = 0 \qquad \text{and} \qquad K = \mathrm{const} \neq \frac{\Lambda}{3}$$

Corollary: There are no rotating solutions

The Petrov type D equation on topological spheres.

$$S = S_2 \,, \qquad g, d\omega$$ **axi-symmetric**

Theorem 5:

The family of axisymmetric solutions of the Petrov type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by two numbers *(A,J)*: the area and angular momentum, respectively. They take the following values in $\mathbb{R}^+ \times \mathbb{R}$:

$$\Lambda > 0$$

$$J \in \Big(-\infty, \infty \Big) \text{ for } A \in \Big(0, \frac{12\pi}{\Lambda} \Big) \text{ and } |J| \in \Big[0, \frac{A}{16\pi}\sqrt{\frac{\Lambda \ A}{12\pi} - 1} \Big) \text{ for } A \in \Big(\frac{12\pi}{\Lambda}, \infty \Big)$$

$$\Lambda \leq 0$$

$$J \in \Big(-\infty, \infty \Big) \text{ and } A \in \Big(0, \infty \Big)$$

The embedding in spacetimes

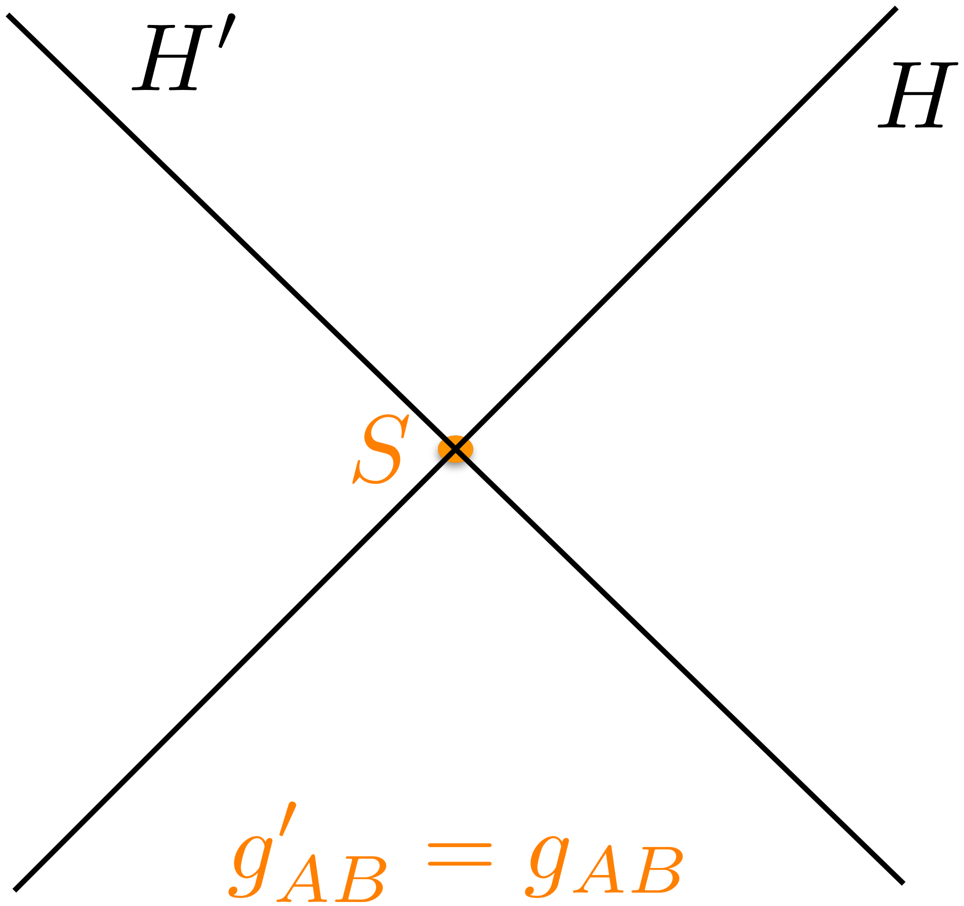
All the horizons obtained via Theorem 5 can be embedded either in non-extremal Kerr-(anti) de Sitter spacetime or in so called near horizon geometry spacetime obtained by the near extremal horizon limit. The embedding preserves intrinsic geometry of each horizon:

$$g_{ab}, \nabla_a$$

The existence problem for non-axisymmetric solutions on topologically spherical S is open.

A bifurcated Petrov type D horizon: data

suggested by Racz 2018



$$g'_{AB} = g_{AB}$$

$$\omega'_A = -\omega_A$$

$$\Psi' = \bar{\Psi}$$

A bifurcated Petrov type D horizon: equations

The Petrov type D equations

for H:

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

and for *H':*

$$m^A m^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

hold simultaneously on S $\Rightarrow$ additional (axial) symmetry

M.J. Cole, I. Rácz, J.A. Valiente Kroon 2018

JL, A. Szereszewski 2018

In conformally flat coordinates

$$g_{AB}dx^A dx^B = \frac{2}{P^2}dz d\bar{z} \qquad m^A\partial_A = P\partial_z$$

$$\partial_{\bar{z}}(P^2\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{F(z)}{P^2}$$

$$\partial_z(P^2\partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{\bar{G}(\bar{z})}{P^2}$$

$$\partial_z\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \partial_{\bar{z}}\partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \quad \Rightarrow \quad \partial_z\left(\frac{F(z)}{P^2}\right) = \partial_{\bar{z}}\left(\frac{\bar{G}(\bar{z})}{P^2}\right)$$

$$\Rightarrow \quad \mathcal{L}_\Phi g_{AB} = 0 \qquad \Phi := F(z)\partial_z - \bar{G}(\bar{z})\partial_{\bar{z}}$$

$$\Rightarrow \quad \mathcal{L}_\Phi\, d\omega = 0 \qquad = X^{(1,0)} - X^{(0,1)}$$

The axial symmetry without the rigidity theorem

Theorem 6:

Suppose (g_{AB}, ω_A) defined on S satisfy the Petrov type D equation

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

and the conjugate one

$$m^A m^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Then, there is a vector field Φ at S such that

$$\mathcal{L}_\Phi g_{AB} = 0 \qquad \textbf{and} \qquad \mathcal{L}_\Phi d\omega = 0$$

$$\Phi^A = \mathrm{Re}/\mathrm{Im}\left(d\mathrm{Area}^{AB} \partial_A (\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \right)$$

Corollary: the axial symmetry on $S = S_2$

Application to the NHG equation

Lemma:

Suppose (g_{AB}, ω_A) **are defined on a compact 2d** S

and satisfy the NHG equation:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0 \ .$$

Then $\Psi + \frac{1}{6}\Lambda \neq 0$

Corollary: (g_{AB}, ω_A) **also satisfy the Petrov type D equation:**

$$\bar{m}^A\bar{m}^B\nabla_A\nabla_B(\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Application to the NHG equation

Theorem 7:

Suppose (g_{AB}, ω_A) **are defined on a compact 2d** S

and satisfy the NHG equation:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0 \ ;$$

Suppose

$$\chi_E(S) \le 0$$

Then $$K = \Lambda \le 0, \qquad \omega_A = 0$$

Dobkowski-Ryłko, Kamiński, JL, Szereszewski

Summary

- The type D equation:

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(-\frac{1}{2}K - \frac{1}{2} i\mathcal{O} + \frac{\Lambda}{6} \right)^{-\frac{1}{3}} = 0$$

- Non-twisting of the second double principal vector if:

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B + \frac{1}{2}(\Lambda - K) g_{AB} = 0$$

- All the axisymmetric solutions of the type D eq. on topological sphere parametrized by (A, J);

- All solutions on genus>0 derived (non-rotating);

- The extra (axial) symmetry in the case of bifurcated horizon;

- Open problems: existence of non-axisymmetric solutions on topological sphere

Complete solution of the NHG equation for non-zero genus

Thank you