The Brans-Dicke cosmology with non-minimally coupled scalar field

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Working cosmological model:

"BB" $\rightarrow$ Inflation $\rightarrow$ RDE $\rightarrow$ MDE $\rightarrow$ de Sitter

Dynamical system theory:

an unstable node $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a stable node/attractor

or without BB singularity

an unstable node $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a stable node/attractor

GR with ξ

We start from the total action of the theory

$$S = S_g + S_\psi \,,$$

consisting of the gravitational part described by the standard Einstein-Hilbert action integral

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,R\,,$$

where $\kappa^2 = 8\pi G$, and the matter part of the theory is in the form of non-minimally coupled scalar field

$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big(\varepsilon\nabla^\alpha\psi\,\nabla_\alpha\psi + \varepsilon\xi R\psi^2 + 2U(\psi)\Big)\,,$$

where $\varepsilon = +1, -1$ corresponds to the canonical and the phantom scalar field, respectively.

An asymptotically quadratic potential function

$$U(\phi) = \pm\frac{1}{2}m^2\psi^2 \pm M^{4+n}\psi^{-n}\,,$$

where $n > -2$
projective coordinates

$$u = \frac{\dot{\psi}}{H\psi} = \frac{\mathrm{d}\ln\psi}{\mathrm{d}\ln a}\,, \quad v = \frac{6}{\kappa^2\psi^2}\,,$$

the energy conservation condition

$$\left(\frac{H}{H_0}\right)^2 = \mu\frac{1+\beta v^{1+\frac{n}{2}}}{v - \varepsilon(1-6\xi)u^2 - \varepsilon 6\xi(u+1)^2}$$

where

$$\mu = \pm\frac{m^2}{H_0^2}\,, \quad \beta = \pm 2\left(\frac{\kappa^2}{6}\right)^{1+\frac{n}{2}}\frac{M^{4+n}}{m^2}\,,$$

Physics from dynamics

$$u^{*} = -\frac{2\xi}{1-4\xi}\,, \qquad v^{*} = 0$$

eigenvalues

$$\lambda_1 = -4 + \frac{1}{1-4\xi}\,, \qquad \lambda_2 = \frac{4\xi}{1-4\xi}$$

an unstable node

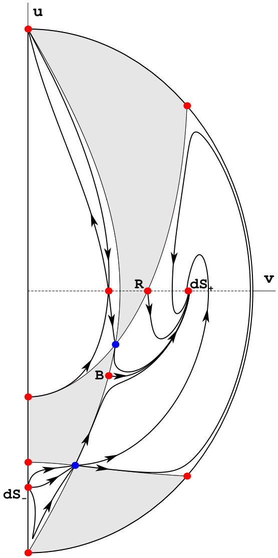
$$\frac{3}{16} < \xi < \frac{1}{4}$$

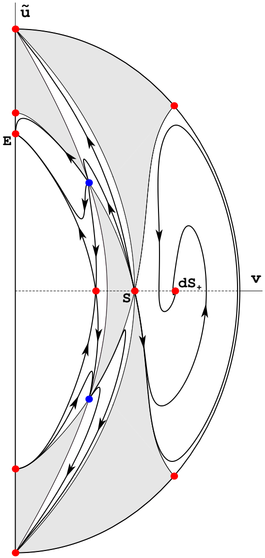
the energy conservation condition

$$\left(\frac{H(0)}{H(a_0)}\right)^2\Bigg|^{*} = -\varepsilon\mu\frac{(1-4\xi)^2}{2\xi(1-6\xi)(3-16\xi)}\,,$$

subject to condition $-\varepsilon\left(\pm\frac{m^2}{H_0^2}\right) > 0\,.$

Non-minimally coupled scalar field





The Brans-Dicke cosmology with ξ

The total action of the theory is

$$S = S_g + 16\pi S_\psi \,,$$

consisting of the gravitational part described by the action integral

$$S_g = \int \mathrm{d}^4 x \sqrt{-g}\Big(\phi R - \frac{\omega_{\mathrm{BD}}}{\phi}\nabla^\alpha\phi\nabla_\alpha\phi - 2V(\phi)\Big)\,,$$

where ω_{BD} is a dimensionless parameter of the gravitational theory and $\phi = M^2_{\mathrm{Pl,eff}} = \frac{1}{G_{\mathrm{eff}}}$. The matter part of the theory is in the form of non-minimally coupled scalar field

$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4 x \sqrt{-g}\Big(\varepsilon\nabla^\alpha\psi\,\nabla_\alpha\psi + \varepsilon\xi R\psi^2 + 2U(\psi,\phi)\Big)\,,$$

where $\varepsilon = +1, -1$ corresponds to the canonical and the phantom scalar field, respectively, and ξ is a dimensionless parameter of the substantial part of the theory.

Dynamics in dimensionless dynamical variables

The potential functions for the scalar fields

$$V(\phi) = \Lambda\phi\,, \qquad U(\psi,\phi) = \pm\frac{1}{2}m^2\psi^2 \pm M^{2(2-m)}\phi^m \left(\frac{\psi^2}{\phi}\right)^{-n}.$$

Dimensionless dynamical variables

$$x = \frac{\dot{\phi}}{\phi H} = \frac{\mathrm{d}\ln\phi}{\mathrm{d}\ln a}\,, \quad u = \frac{\dot{\psi}}{\psi H} = \frac{\mathrm{d}\ln\psi}{\mathrm{d}\ln a}\,, \quad \bar{v} = \frac{3}{4\pi}\frac{\phi}{\psi^2}\,, \quad w = \left(\frac{\phi}{\phi_0}\right)^{m-1}\bar{v}^n\,.$$

Dimensionless parameters

$$\mu = \pm\frac{m^2}{H_0^2}\,, \quad \alpha = \pm\frac{\Lambda}{3m^2}\,, \quad \beta = \pm 2\left(\frac{4\pi}{3}\right)^{n+1}\frac{M^{2(2-m)}}{m^2\phi_0^{1-m}}\,.$$

Dynamics in dimensionless dynamical variables

The energy conservation condition, i.e. the Hubble function

$$\left(\frac{H(a)}{H(a_0)}\right)^2 = \mu \frac{\alpha\bar{v} + 1 + \beta w \bar{v}}{\bar{v}\left(1 - \frac{\omega_{\text{BD}}}{6}x^2 + x\right) - \varepsilon\left((1-6\xi)u^2 + 6\xi(u+1)^2\right)},$$

and the acceleration equation

$$\frac{\dot{H}}{H^2} = -2 + \frac{1}{\frac{3+2\omega_{\text{BD}}}{2\omega_{\text{BD}}}\bar{v} - \varepsilon 6\xi(1-6\xi)}\Bigg\{ -\frac{3+2\omega_{\text{BD}}}{2\omega_{\text{BD}}}x^2\bar{v} - \varepsilon(1-6\xi)u^2 +$$

$$+\frac{2}{\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\Bigg(\left(1 + \frac{3}{4\omega_{\text{BD}}}\right)\alpha\bar{v} + (1-3\xi) +$$

$$+\left(1 + 3\xi n + \frac{3}{4\omega_{\text{BD}}}(m+n)\right)\beta w\bar{v}\Bigg)\Bigg\}.$$

Dynamical system

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\ln a} &= -\frac{1}{2}x(x+2) - \left(x - \frac{3}{\omega_{\mathrm{BD}}}\right)\left(\frac{\dot{H}}{H^2}+2\right) - \\
&\quad - \frac{3}{\omega_{\mathrm{BD}}\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\left(\alpha + (m+n)\beta w\right), \\
\frac{\mathrm{d}u}{\mathrm{d}\ln a} &= -u(u+1) - (u+6\xi)\left(\frac{\dot{H}}{H^2}+2\right) - \\
&\quad - \frac{\varepsilon}{\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\left(1 - n\beta w\bar{v}\right), \\
\frac{\mathrm{d}\bar{v}}{\mathrm{d}\ln a} &= -2\bar{v}\left(u - \frac{1}{2}x\right), \\
\frac{\mathrm{d}w}{\mathrm{d}\ln a} &= -2w\left(nu - \frac{1}{2}(m+n-1)x\right).
\end{aligned}
$$

Asymptotic de Sitter state

Critical points

$$x^*_{1,2} = -3 \pm 3\sqrt{\frac{2}{3\omega_{\mathrm{BD}}}\left(\frac{4+3\omega_{\mathrm{BD}}}{2} + \varepsilon\alpha\frac{2\xi(1-6\xi)(3-16\xi)}{(1-4\xi)^2}\right)}\,,$$

$$u^* = -\frac{2\xi}{1-4\xi}\,, \quad \bar{v}^* = 0\,, \quad w^* = 0\,.$$

The energy conservation condition and the acceleration equation

$$\left(\frac{H(a)}{H(a_0)}\right)^2\Bigg|^* = -\varepsilon\mu\frac{(1-4\xi)^2}{2\xi(1-6\xi)(3-16\xi)} > 0\,,$$

$$\frac{\dot{H}}{H^2}\Bigg|^* = 0\,.$$

(In)Stability of the de Sitter state

The eigenvalues of the linearisation matrix

$$\lambda_1 = -3 - x^* , \quad \lambda_2 = -\frac{3-16\xi}{1-4\xi} ,$$
$$\lambda_3 = \frac{4\xi}{1-4\xi} + x^* , \quad \lambda_4 = \frac{4\xi}{1-4\xi} n + (m+n-1)x^* .$$

The asymptotic state is unstable with respect to expansion of the Universe iff

$$\lambda_i > 0 .$$

Conformal Invariance and extra dimensions

For the non-minimally coupled scalar field with the potential function $U(\psi) = U_0\psi^p$:

$$\xi = \xi_{\mathrm{conf}} = \frac{1}{4}\frac{D-2}{D-1}\,, \qquad p = p_{\mathrm{conf}} = \frac{2D}{D-2}\,,$$

For the Brans-Dicke scalar field with the potential function $V(\phi) = V_0\phi^q$:

$$\omega_{\mathrm{BD}} = -\frac{D-1}{D-2}\,, \qquad q = q_{\mathrm{conf}} = \frac{D}{D-2}\,.$$

In this way we obtain a discrete set of theoretically allowed values of the dimensionless coupling constants of the theory suggested by the conformal invariance condition of the scalar fields in $D \geq 2$ space-time dimensions:

$$\{(D, \xi, p, \omega_{\mathrm{BD}}, q)\} = \Big\{ (2, 0, \infty, -\infty, \infty,)\,, \left(3, \frac{1}{8}, 6, -2, 3\right), \left(4, \frac{1}{6}, 4, -\frac{3}{2}, 2\right), \left(5, \frac{3}{16}, \frac{10}{3}, -\frac{4}{3}, \frac{5}{3}\right), \left(6, \frac{1}{5}, 3, -\frac{5}{4}, \frac{3}{2}\right), \ldots, \left(\infty, \frac{1}{4}, 2, -1, 1\right) \Big\}.$$

(In)Stability of the de Sitter state

$$\mu = \pm \frac{m^2}{H_0^2}\,, \quad \alpha = \pm \frac{\Lambda}{m^2}$$

For $m = 1$ the initial de Sitter state in unstable with respect to the expansion

$$\frac{3}{16} < \xi < \frac{1}{4}\,, \quad n > 0\,, \quad \varepsilon\mu < 0\,, \quad -\frac{1}{6\xi} < \varepsilon\alpha < \frac{(1-4\xi)^2}{16\xi^2(1-6\xi)}\,.$$

$$U(\psi,\,\phi) = \pm\frac{1}{2}m^2\psi^2 \pm M^2\phi^{1+n}\psi^{-2n}$$

For $n = -2$ the initial de Sitter state in unstable with respect to the expansion

$$\frac{3}{16} < \xi < \frac{1}{4}\,, \quad m \leq \frac{9-44\xi}{3(1-4\xi)}\,, \quad \varepsilon\mu < 0\,, \quad -\frac{1}{6\xi} < \varepsilon\alpha < \frac{(1-4\xi)^2}{16\xi^2(1-6\xi)}\,.$$

$$U(\psi,\,\phi) = \pm\frac{1}{2}m^2\psi^2 \pm M^{2(2-m)}\phi^{m-2}\psi^4$$

A new dynamical GR limit

The equation of motion for the effective gravitational coupling constant

$$\frac{\mathrm{d}x}{\mathrm{d}\ln a} = -\frac{1}{2}x(x+2) - \left(x - \frac{3}{\omega_{\mathrm{BD}}}\right)\left(\frac{\dot{H}}{H^2}+2\right) - \\ - \frac{3}{\omega_{\mathrm{BD}}\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\left(\alpha + (m+n)\beta w\right),$$

GR limit: $\phi = \mathrm{const.} = M_{Pl}^2 = \frac{1}{G_N} \quad \Rightarrow \quad x = \frac{\mathrm{d}\ln\phi}{\mathrm{d}\ln a} = 0 \quad \& \quad \frac{\mathrm{d}x}{\mathrm{d}\ln a} = 0\,.$

$$\frac{3}{\omega_{\mathrm{BD}}}\left(\frac{\dot{H}}{H^2} + 2 - \frac{1}{\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\left(\alpha + (m+n)\beta w\right)\right) = 0\,.$$

A new dynamical GR limit

Invariant manifold in 4-dimensional phase space

$$\frac{\dot{H}}{H^2} + 2 - \frac{1}{\mu}\left(\frac{H(a)}{H(a_0)}\right)^{-2}\left(\alpha + (m+n)\beta w\right) = 0$$

can be solved for one of the dynamical variables

$$w = w(u, \bar{v})\,.$$

Conclusions

- In Brans-Dicke cosmology with non-minimally coupled scalar field with $\frac{3}{16} < \xi < \frac{1}{4}$ there exists the initial de Sitter state, with $G_{\text{eff}} \to 0$, giving rise to non-singular beginning of the Universe.
- In dynamics of the cosmological model there exists an invariant, asymptotically stable, manifold corresponding to the standard GR.
- Conformal invariance might be the fundamental symmetry both for the gravitational and the substantial part of a cosmological theory...

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- In Brans-Dicke cosmology with non-minimally coupled scalar field with $\frac{3}{16} < \xi < \frac{1}{4}$ there exists the initial de Sitter state, with $G_{\text{eff}} \to 0$, giving rise to non-singular beginning of the Universe.
- In dynamics of the cosmological model there exists an invariant, asymptotically stable, manifold corresponding to the standard GR.
- Conformal invariance might be the fundamental symmetry both for the gravitational and the substantial part of a cosmological theory but only in presence of extra dimensions.