The Penrose inequality for an axially perturbed Schwarzschild black hole

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joint work with J. Kopiński

The Penrose inequality

Area of the Schwarzschild horizon

$$A = 4\pi(2M)^2 = 16\pi M^2$$

hence

$$M = \sqrt{\frac{A}{16\pi}}\ .$$

For the Kerr horizon

$$A = 8\pi M(M + \sqrt{M^2 - a^2})$$

hence (the Penrose inequality)

$$M \geq \sqrt{\frac{A}{16\pi}}\ .$$

The Cauchy data with a horizon

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- a future singularity is surrounded by the event horizon (physically realistic data, the cosmic censorship conjecture)
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then the no-hair theorem etc. (almost) imply that

- the end state is the Kerr metric with M_∞ and A_∞
- $M_\infty \leq M$ (because of radiation)
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Conclusion: the Penrose inequality should be satisfied on the initial surface

Vacuum initial data with a horizon

Constraints on g_{ij}, K_{ij}

$$\nabla_i\left(K^{ij} - g^{ij}H\right) = 0$$
$$R + H^2 - K^2 = 0$$

where $H = K^i{}_i$ and $K^2 = K_{ij}K^{ij}$.

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Marginal outer trapped surface (MOTS) with unit normal vector n^i and mean curvature $h = \nabla_i n^i$:

$$\theta_+ = H - K_{nn} + h = 0$$

$$\theta_- = H - K_{nn} - h \leq 0 \ .$$

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If $H = h = 0$ then the Penrose inequality follows from the Hamiltonian constraint (Geroch, ..., Huisken and Ilmanen).

The Lichnerowicz equation

Let $H = \nabla_i K^{ij} = 0$ and S_0 be a closed surface. Then

$$g'_{ij} = \psi^4 g_{ij}, \quad K'_{ij} = \psi^{-2} K_{ij} \ , \ \ \psi > 0$$

satisfy the constraint equations if ψ satisfies the Lichnerowicz equation

$$\triangle\psi = \frac{1}{8} R\psi - \frac{1}{8} K^2 \psi^{-7} \ .$$

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If, in addition, ψ satisfies the boundary condition on S_0

$$n^i \partial_i \psi + \frac{1}{2} h\psi - \frac{1}{4} K_{nn} \psi^{-3} = 0$$

then S_0 becomes MOTS (D. Maxwell, extra conditions required).

Small perturbations of the Schwarzschild data

As the preliminary data we take

- the Schwarzschild initial data on $t = const$

$$g = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

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The Penrose inequality, if true, should follow from the Lichnerowicz equation and the boundary condition.

Approximation up to K^2 : $\psi = 1 + \psi_1 + \psi_2$

$$\triangle\psi_1 = 0 \ , \quad n^i\partial_i\psi_1 = \frac{1}{4}K_{nn} \ on \ S_0 \tag{1}$$

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Total energy (after the conformal transformation):

$$E = M - \frac{1}{2\pi}\int_{S_\infty}\partial_r\psi d\sigma.$$

Area of the horizon:

$$A = \int_{S_0}\psi^4 d\sigma.$$

An integration of the Lichnerowicz equation yields

$$E = M + \frac{1}{16\pi}\int_{S_0}^{\infty} K^2 dV - \frac{1}{8\pi}\oint_{S_0} K_{nn}(1 - 3\psi_1)d\sigma$$

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The Penrose inequality is equivalent to

$$\int_{2M}^{\infty} r^2\langle K^2\rangle dr + 8M^3\langle K_{nn}\rangle^2 - 12M\langle\psi_1^2\rangle \geq 0 \tag{3}$$

where $\langle ... \rangle$ denotes integral over the unit sphere and ψ_1 depends on K_{nn} via the Neumann problem.

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Problem

Given $K_{nn} \geq 0$ on S_0 what is minimal value of the first integral in (3).

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A convenient form of Q:

$$Q = \sin\theta q_{,\theta} - a\cos\theta \ , \quad a = const$$

where q is regular and $q_{,r}$ is bounded at ∞.

Expansion into Legendre polynomials $P_n(z)$, $z = \cos\theta$:

$$q = \Sigma_{n=1} q_n(r) P_n \ .$$

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Miracle No 1

$$K_{\theta\theta} + \frac{1}{2} r^2 K_{rr} = \Sigma_{n=2} \big((r q_{n,r})_{,r} - \frac{1}{2} n(n+1) \frac{q_n}{r} \big) \tilde{P}_n \ ,$$

where polynomials

$$\tilde{P}_n = \frac{1}{n+2} (n P_n - \frac{2}{n-1} P_{n-1,z})$$

are orthogonal.

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Hence $\langle K^2 \rangle = \Sigma F_n(q_n)$, $F_n \geq 0$ and we can look separately for a minimum of each integral

$$\int_{2M}^{\infty} r^2 F_n dr.$$

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First (to my knowledge) proof with horizon which is not a minimal surface.

Generalizations: nonsymmetric K, nonspherical horizons (?)