NONTRIVIAL PROPERTIES OF MOMENTUM SPACE AND RELATIVE LOCALITY IN K-POINCARE AND K-DE SITTER ALGEBRAS

Giulia Gubitosi

The 5th Conference of the Polish Society on Relativity - September 2018

Non-quantum limit of quantum gravity and Deformed Special Relativity

- $\hbar \to 0$ regime of Quantum Gravity

  if the limit is taken keeping $\frac{\hbar}{G} = const$, QG effects are governed by an energy scale

$$E_P = \sqrt{\frac{\hbar c^5}{G}} \to const \qquad \ell_P = \sqrt{\frac{\hbar G}{c^3}} \to 0$$

- Because we can define an energy scale but not a length scale, it is natural to look at physics from the point of view of momentum space rather than spacetime (*focus on relativistic symmetries*)

- We can construct a deformation of the Poincaré algebra, such that the energy scale becomes a second relativistic invariant besides the speed of light

  •*Amelino-Camelia, IJMPD 2002, PLB 2001*
  •*Kowalski-Glikman, IJMPA 2001*
  •*Magueijo, Smolin, PRL 2002, PRD 2003*

- Further motivations:

  - Indications that the effective action of matter coupled to 2+1 quantum gravity describes matter fields subject to deformed Poincaré symmetries
    (see talk by G. Rosati later today)

    •*Freidel, Kowalski-Glikman, Smolin, PRD 2004*
    •*Freidel, Livine PRL 2006*
    •*Cianfrani, Kowaslki-Glikman, Pranzetti, Rosati, PRD 2016*

  - Indications that the spacetime symmetries emerging in the Minkowski regime of LQG are described by a deformed Poincaré group

    •*Bojowald, Paily, PRD 2013*
    •*Amelino-Camelia, da SIlva, Ronco, Cesarini, Lecian PRD2017*
    •*Brahma, Ronco, Amelino-Camelia, Marciano, PRD2017*
    •*Brahma, Ronco, PLB 2018*

Poisson-Hopf algebra description of relativistic symmetries

✦ Hopf algebras provide a consistent mathematical framework to deform special-relativistic symmetries and introduce an invariant energy scale

- *J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B 1992*
- *S. Majid, H. Ruegg, Phys. Lett. B 1994*
- *J. Kowalski-Glikman, S. Nowak 2002-2003*

✦ in the semiclassical approximation, the symmetries of phase space are described by Poisson brackets satisfying the same relations as the commutators of the Hopf algebra

✦ k-Poincaré is the most used Hopf algebra to develop phenomenology associated to deformed Poincaré symmetry, in particular focussing on energy-dependent time of travel of relativistic particles

- *Amelino-Camelia, Kowalski-Glikman, Mandanici, Procaccini, Int. J. Mod. Phys. A20 (2005)*

✦ opportunities for phenomenology arise for example in the study of the propagation of very high energy particles (photons, neutrinos) from astrophysical sources
(see talk by G. Amelino-Camelia earlier today)

- *Amelino-Camelia, Ellis, Mavromatos, Nanopoulos, Sarkar, Nature 393 (1998)*
- *M. Ackermann et al. (Fermi GBM/LAT), Nature 462(2009)*
- *Xu, Ma, Astropart.Phys. 82 (2016)*
- *Amelino-Camelia, D’Amico, Rosati, Loret, Nat.Astron. 1 (2017)*

k-Poincaré Poisson-Hopf algebra

- algebra of symmetries in bicrossproduct coordinates (1+1 dimensions)

$$\{\mathcal{P}_1, \mathcal{P}_0\} = 0$$
$$\{\mathcal{N}, \mathcal{P}_0\} = \mathcal{P}_1$$
$$\{\mathcal{N}, \mathcal{P}_1\} = \frac{1 - e^{-2\ell\mathcal{P}_0}}{2\ell} - \frac{\ell}{2}\mathcal{P}_1^2$$

$$\left[\ell = \frac{1}{\kappa} \sim \frac{1}{E_p}\right]$$

- first Casimir

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\sinh\left(\frac{\ell\mathcal{P}_0}{2}\right)\right)^2 - e^{\ell\mathcal{P}_0}\mathcal{P}_1^2$$

- coproducts and antipodes

$$\Delta(\mathcal{P}_0) = \mathcal{P}_0 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_0 \qquad S(\mathcal{P}_0) = -\mathcal{P}_0$$
$$\Delta(\mathcal{P}_1) = \mathcal{P}_1 \otimes \mathbb{I} + e^{-\ell\mathcal{P}_0} \otimes \mathcal{P}_1 \qquad S(\mathcal{P}_1) = -e^{\ell\mathcal{P}_0}\mathcal{P}_1$$
$$\Delta(\mathcal{N}) = \mathcal{N} \otimes \mathbb{I} + e^{-\ell\mathcal{P}_0} \otimes \mathcal{N} \qquad S(\mathcal{N}) = -e^{\ell\mathcal{P}_0}\mathcal{N}$$

•*Lukierski, Nowicki, Ruegg, Phys. Lett. B 293 (1992)*
•*Lukierski, Ruegg, Nowicki, Tolstoi, Phys. Lett. B 264 (1991)*
•*Majid, Ruegg, Phys.Lett. B334 (1994)*

k-Poincaré representation on momentum space

- because spacetime translations close a subalgebra, they can be represented as an algebra of functions over momentum space

•Kowalski-Glikman, Nowak, CQG 2003
•Gubitosi, Mercati, CQG 2013

- correspondence between structures of the Hopf sub-algebra and of the momentum space:

translations $P_\mu(p)$	coordinates over manifold p_μ
change of basis of the algebra	diffeomorphism
coproduct map $\Delta P_\mu(p,q)$	composition law of momenta $(p \oplus q)_\mu$
antipode $S(P_\mu)(p)$	inversion $(\ominus p)_\mu$
coassociativity $(\Delta \otimes \mathtt{Id}) \circ \Delta = (\mathtt{Id} \otimes \Delta) \circ \Delta$	associativity of composition rule $(p \oplus q) \oplus k = p \oplus (q \oplus k)$

Geometric properties of the k-Poincaré momentum space manifold

- k-Poincaré momenta live on a (portion of) de Sitter manifold

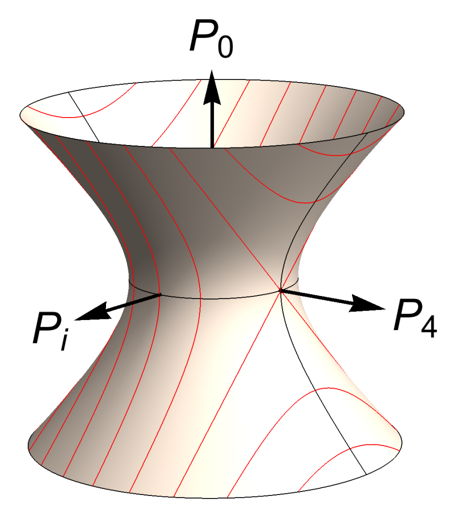
•*Kowalski-Glikman, Nowak, CQG 2003*
•*Kowalski-Glikman PLB 2002*
•*Gubitosi, Mercati, CQG 2013*
•*Amelino-Camelia, Arzano, Kowalski-Glikman, Rosati, Trevisan, CQG 2012*

change to a basis where the algebra is trivial (coproducts still nontrivial)

$$P_0(p_0,p_1) = \frac{\sinh(\ell p_0)}{\ell} + \frac{\ell p_1^2}{2} e^{\ell p_0}$$
$$P_1(p_0,p_1) = p_1 e^{\ell p_0}$$
$$P_4(p_0,p_1) = \frac{\cosh(\ell p_0)}{\ell} - \frac{\ell p_1^2}{2} e^{\ell p_0}$$

these new generators satisfy the relation

$$P_0^2 - P_1^2 - P_4^2 = -\frac{1}{\ell^2}$$



defining relation of a 1+1 dimensional de Sitter manifold embedded in a 2+1 Minkowski manifold

the energy scale is playing a crucial role in the geometry of momentum space, since it defines its radius of curvature

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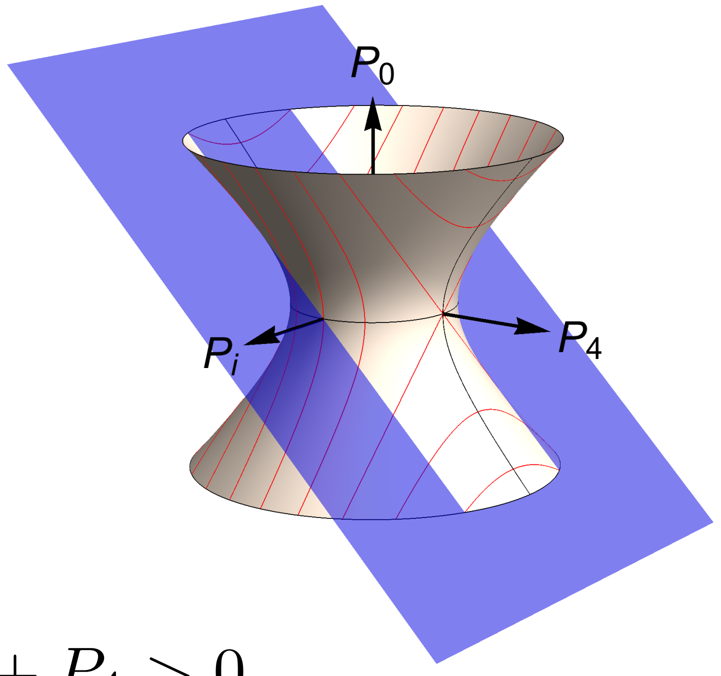

defining relation of a 1+1 dimensional de Sitter manifold embedded in a 2+1 Minkowski manifold

the energy scale is playing a crucial role in the geometry of momentum space, since it defines its radius of curvature

- bicrossproduct coordinates only cover half of the manifold: $P_0 + P_4 > 0$

Curved momentum space and kinematics of free particles

- the first Casimir of the algebra gives the mass-shell condition

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\sinh\left(\frac{\ell\mathcal{P}_0}{2}\right)\right)^2 - e^{\ell\mathcal{P}_0}\mathcal{P}_1^2 \longrightarrow m^2 = \left(\frac{2}{\ell}\sinh\left(\frac{\ell p_0}{2}\right)\right)^2 - e^{\ell p_0}p_1^2$$

in the massless case : $p_1(p_0) = \dfrac{1 - e^{-\ell p_0}}{\ell}$

- the dispersion relation of free particles is invariant under boosts

$$[\mathcal{C}_\ell, \mathcal{N}] = 0 \longrightarrow \left(\frac{2}{\ell}\sinh\left(\frac{\ell p'_0}{2}\right)\right)^2 - e^{\ell p'_0}p_1'^2 = m^2$$

Curved momentum space and kinematics of free particles

✦ the first Casimir of the algebra gives the mass-shell condition

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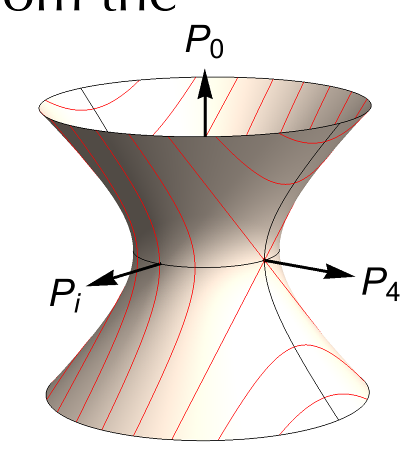
✦ the dispersion relation of free particles is invariant under boosts

$$[\mathcal{C}_\ell, \mathcal{N}] = 0 \longrightarrow \left(\frac{2}{\ell}\sinh\left(\frac{\ell p_0'}{2}\right)\right)^2 - e^{\ell p_0'}p_1'^2 = m^2$$

✦ from the point of view of the momentum space:

the dispersion relation is given by the curves of constant geodesic distance from the origin of momentum space



invariance of the dispersion relation is due to the invariance of the line element $ds_p^2 = dp_0^2 - e^{2\ell p_0}dp_1^2$

$$\begin{aligned} p_0' &= p_0 + \xi p_1 \\ p_1' &= p_1 + \xi\left(\frac{1-e^{-2\ell p_0}}{2\ell} - \frac{\ell}{2}p_1^2\right) \end{aligned} \longrightarrow (ds_p^2)' \equiv dp_0'^2 - e^{2\ell p_0'}dp_1'^2 = ds_p^2$$

(boosts are isometries of the k-Poincaré momentum space)

k-Poincaré particle kinematics

✦ spacetime is defined via a classical phase-space construction (no quantum effects in the relative locality limit) - coordinates are the objects that define a trivial symplectic structure together with momenta:

$$\{p_1, p_0\} = 0$$
$$\{x^1, x^0\} = 0$$
$$\{x^\mu, p_\nu\} = \delta^\mu_\nu$$

✦ representation of the algebra of symmetries on phase space

$$\mathcal{P}_0 = p_0$$
$$\mathcal{P}_1 = p_1$$
$$\mathcal{N} = p_1 x^0 + \left(\frac{1 - e^{-2\ell p_0}}{2\ell} - \frac{\ell}{2}p_1^2\right) x^1$$

✦ evolution of phase space coordinates is given by the a Hamiltonian construction

$$\dot{x}^0 = \{\mathcal{C}_\ell, x^0\} = \frac{1}{\ell}\left(e^{\ell p_0} - e^{-\ell p_0}\right) - \ell p_1^2 e^{\ell p_0}$$
$$\dot{x}^1 = \{\mathcal{C}_\ell, x^1\} = 2\, p_1\, e^{\ell p 0}.$$

✦ massless coordinate velocity depends on the energy of the particle: $v \equiv \frac{\dot{x}^1}{\dot{x}^0} = -e^{\ell p_0}$

massless particle worldline: $x^1 - \bar{x}^1 = -e^{\ell p 0}(x^0 - \bar{x}^0)$

✦ same time delays obtained with 'k-Minkowski coordinates' if accounting for the deformed action of translations upon them

$$\chi^1 = x^1$$
$$\chi^0 = x^0 - \ell x^1 p_1$$

$$\{\chi^0, \chi^1\} = \ell \chi^1$$
$$\{\chi^0, p_1\} = -\ell p_1$$
$$\{\chi^0, p_0\} = 0$$

•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*
•*Gubitosi, Barcaroli, PRD 93 (2016)*

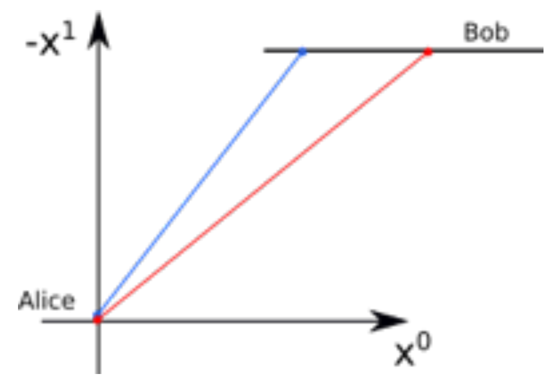
Particle trajectories and relativity of locality

worldlines of two massless particles emitted simultaneously with different energies

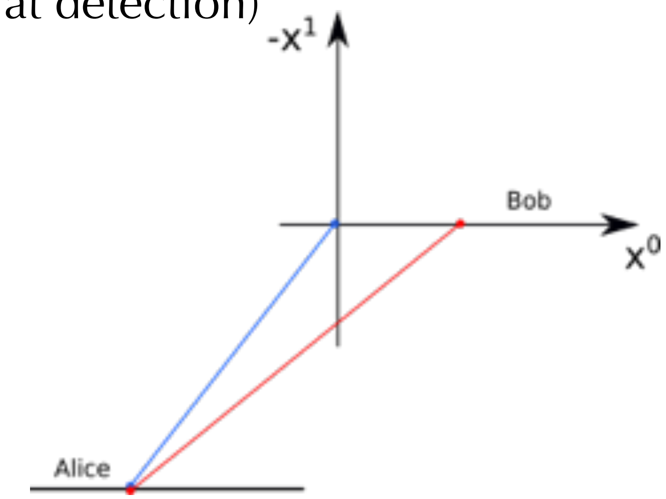
worldlines seen by Alice (local at emission)

worldlines seen by Bob (local at detection)

✦ using coordinates dual to momenta:

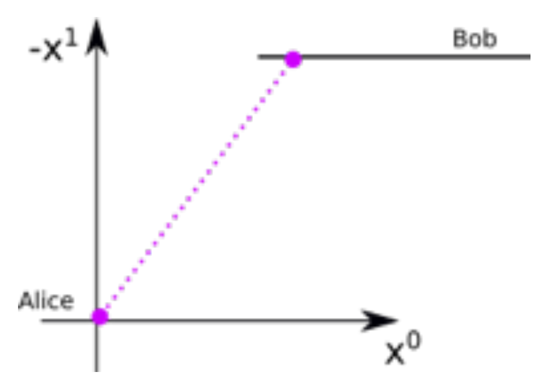


$$x^1_A = -e^{\ell p_0} x^0_A$$
$$x^1_A = -e^{\ell \tilde{p}_0} x^0_A$$

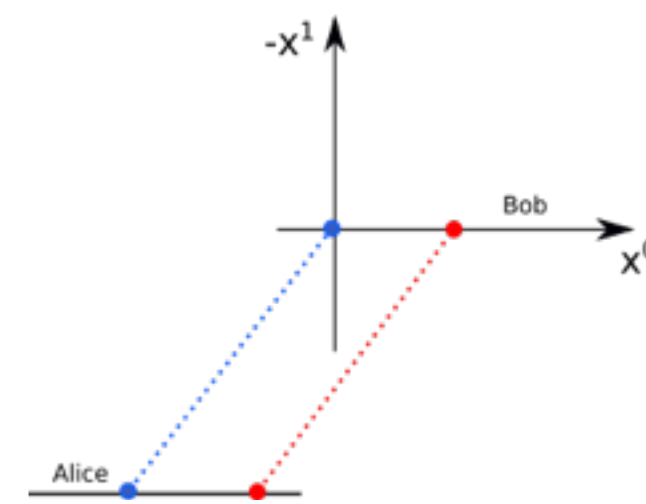


$$\begin{aligned} x^0_B &= \mathcal{T}_a \triangleright x^0_A = x^0_A - a^0 \\ x^1_B &= \mathcal{T}_a \triangleright x^1_A = x^1_A - a^1 \end{aligned}$$

✦ using ‘k-Minkowski coordinates’:



$$\chi^1 - \bar{\chi}^1 = \chi^0 - \bar{\chi}^0$$



$$\chi^0_B = \mathcal{T}_a \triangleright \chi^0_A = \chi^0_A - a^0 + a^1 \ell p_1$$
$$\chi^1_B = \mathcal{T}_a \triangleright \chi^1_A = \chi^1_A - a^1$$

✦ measurements done <u>locally</u> (i.e. at spatial origin of each observer) do not depend on choice of coordinates: Alice emits the particles at the same time and Bob detects then with a time delay

$$\Delta x^0 = a^0 \left(e^{-\ell \Delta p_0} - 1\right)$$

•*Amelino-Camelia, Loret, Rosati, PLB 2011*
•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, CQG 2013*

de Sitter spacetime - symmetries, phase space and particle kinematics

- line element in comoving coordinates

$$ds^2 = (dx^0)^2 - e^{2Hx^0}(dx^1)^2$$

- algebra of symmetries (co-algebra sector is trivial)

$$\begin{aligned} \{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1 \\ \{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N} \\ \{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0 \end{aligned}$$

- mass Casimir $\quad \mathcal{C}_{dS} = \mathcal{P}_0^2 - \mathcal{P}_1^2 + 2H\mathcal{N}\mathcal{P}_1$

- representation of symmetry generators:

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0\,, \\ \{x^\mu, p_\nu\} &= -\delta^\mu_\nu\,, \\ \{p_\mu, p_\nu\} &= 0\,. \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathcal{P}_0 &= p_0 - Hx^1 p_1 \\ \mathcal{P}_1 &= p_1 \\ \mathcal{N} &= x^1 p_0 + p_1\left(\frac{1-e^{-2Hx^0}}{2H} - \frac{H}{2}(x^1)^2\right) \end{aligned}$$

- the massless condition $\mathcal{C}_{dS} = 0$ relates energy and spatial momentum, encoding energy redshift

$$p_0 = |p_1| e^{-Hx^0}$$

- particles worldline

$$\begin{aligned} \dot{x}^1 &= \{\mathcal{C}_{dS}, x^1\} = -2e^{-2Hx^0} p_1 \\ \dot{x}^0 &= \{\mathcal{C}_{dS}, x^0\} = 2p_0 \end{aligned} \quad \longrightarrow \quad x^1(x^0) - \bar{x}^1 \equiv \int_{\bar{x}^0}^{x^0} \frac{\dot{x}^1}{\dot{x}^0} dx^0 = \left(\frac{e^{-H\bar{x}^0} - e^{-Hx^0}}{H}\right)$$

Duality between de Sitter spacetime and de Sitter momentum space

de Sitter spacetime	de Sitter momentum space
spacetime metric $ds^2 = (dx^0)^2 - e^{2Hx^0}(dx^1)^2$	momentum space metric $ds_p^2 = dp_0^2 - e^{2\ell p_0} dp_1^2$
worldline $x^1 = \frac{1 - e^{-Hx^0}}{H}$	dispersion relation $p_1 = \frac{1 - e^{-\ell p_0}}{\ell}$
dispersion relation $p_1 = -e^{Hx^0} p_0$	worldline $x^1 = -e^{\ell p_0} x^0$
generators of translations $\mathcal{P}_1 = p_1$, $\mathcal{P}_0 = p_0 - Hx^1 p_1$	'k-Minkowski coordinates' $\chi^1 = x^1$, $\chi^0 = x^0 - \ell x^1 p_1$

•Amelino-Camelia, Barcaroli, Gubitosi, Loret, CQG 2013

✦ related to the fact that in Hopf algebras noncommutativity induces curvature in the dual space, and viceversa

•Majid ArXiv: hep-th/0604130

Duality between de Sitter spacetime and de Sitter momentum space

- the duality is even more apparent when looking at the algebra of the full phase space

de Sitter spacetime

$$
\begin{aligned}
\{x^\mu, p_\nu\} &= -\delta^\mu_\nu \\[1em]
\{x^1, x^0\} &= 0 \\
\{\mathcal{N}, x^0\} &= x^1 \\
\{\mathcal{N}, x^1\} &= \tfrac{1-e^{-2Hx^0}}{2H} - \tfrac{H}{2}(x^1)^2 \\[1em]
\{p_0, p_1\} &= 0 \\
\{p_0, \mathcal{N}\} &= p_1 e^{-2Hx^0} \\
\{p_1, \mathcal{N}\} &= p_0 - H p_1 x^1 \\[1em]
\{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1 \\
\{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N} \\
\{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0
\end{aligned}
$$

the Casimir of the algebra $\{\mathcal{N}, \mathcal{P}_0, \mathcal{P}_1\}$
determines the dispersion relation

the Casimir of the algebra $\{\mathcal{N}, x^0, x^1\}$
determines the worldline
(and is the Newton-Wigner operator)

de Sitter momentum space

$$
\begin{aligned}
\{p_\nu, x^\mu\} &= -\delta^\mu_\nu \\[1em]
\{\mathcal{P}_1, \mathcal{P}_0\} &= 0 \\
\{\mathcal{N}, \mathcal{P}_0\} &= \mathcal{P}_1 \\
\{\mathcal{N}, \mathcal{P}_1\} &= \tfrac{1-e^{-2\ell\mathcal{P}_0}}{2\ell} - \tfrac{\ell}{2}\mathcal{P}_1^2 \\[1em]
\{x^0, x^1\} &= 0 \\
\{x^0, \mathcal{N}\} &= x^1 e^{-2\ell p_0} \\
\{x^1, \mathcal{N}\} &= x^0 - \ell x^1 p_1 \\[1em]
\{\chi^0, \chi^1\} &= \ell\chi^1 \\
\{\chi^0, \mathcal{N}\} &= \chi^1 - \ell\mathcal{N} \\
\{\chi^1, \mathcal{N}\} &= \chi^0
\end{aligned}
$$

the Casimir of the algebra $\{\mathcal{N}, p_0, p_1\}$
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the Casimir of the algebra $\{\mathcal{N}, \chi^0, \chi^1\}$
determines the worldline
(and is the Newton-Wigner operator)

Putting spacetime and momentum space curvature together

✦ opportunities for phenomenology arise in contexts where spacetime curvature is actually non-negligible (early universe, propagation of photons from Gamma-ray Bursts etc…)

how to implement deformed relativistic transformations over a curved spacetime?

✦ extension of results found in kP to curved spacetime is non-trivial - interplay between effects of curvature in spacetime and in momentum space

•Amelino-Camelia, Smolin, Starodubtsev, Class. Quant. Grav. 21(2004)
•Marciano, Amelino-Camelia, Bruno, Gubitosi, Mandanici, Melchiorri, JCAP 1006 (2010)

✦ geometrically, in curved ST/flat MS models the MS is the cotangent space of the ST manifold at a point; in curved MS/flat ST models ST is the cotangent space of the momentum manifold at a given momentum 'point' - how does this generalise to cases with curvature on both sides of the phase space?

✦ in the context of Hopf algebras, one can study a k-deformation of the de Sitter algebra

• Lukierski, Ruegg, Nowicki and Tolstoi, Phys. Lett. B 264 (1991)
• Ballesteros, Herranz, del Olmo, Santander, J. Phys. A: Math. Gen. 26 (1993)
• Ballesteros, Herranz, del Olmo, Santander, J. Phys. A: Math. Gen. 27 (1994)

Preliminaries: revisiting the k-Poincaré momentum space construction

- algebra in bicrossproduct coordinates (2+1 dimensions) $\left[z = \ell = \dfrac{1}{\kappa}\right]$

$$
\begin{array}{lll}
\{J, P_1\} = P_2, & \{J, P_2\} = -P_1, & \{J, P_0\} = 0, \\
\{J, K_1\} = K_2, & \{J, K_2\} = -K_1, & \{K_1, K_2\} = -J, \\
\{P_0, P_a\} = 0, & \{P_a, P_b\} = 0, & \{K_a, P_0\} = P_a
\end{array}
$$

$$
\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}\left(1 - e^{-2zP_0}\right) + \frac{z}{2}\,P^2\right) - zP_aP_b\,,
$$

- coproducts

$$
\begin{aligned}
\Delta_z(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta_z(P_a) &= P_a \otimes 1 + e^{-zP_0} \otimes P_a, \\
\Delta_z(J) &= J \otimes 1 + 1 \otimes J, \\
\Delta_z(K_a) &= K_a \otimes 1 + e^{-zP_0} \otimes K_a + z\,\epsilon_{abc} P_b \otimes J_c\,.
\end{aligned}
$$

- Poisson algebra dual to translations

$$
\left[X^0, X^i\right] = -z\,X^i, \qquad \left[X^i, X^j\right] = 0
$$

obtained by dualizing the cocommutator map (its form can be read off from the first-order deformation of the coproducts of P) and gives the k-Minkowski spacetime algebra

Preliminaries: revisiting the k-Poincaré momentum space construction

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\{J, K_1\} = K_2, & \{J, K_2\} = -K_1, & \{K_1, K_2\} = -J, \\
\{P_0, P_a\} = 0, & \{P_a, P_b\} = 0, & \{K_a, P_0\} = P_a \\
\multicolumn{3}{l}{\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}\left(1 - e^{-2zP_0}\right) + \frac{z}{2}P^2\right) - zP_aP_b\,,}
\end{array}
$$

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$$
\begin{aligned}
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\Delta_z(P_a) &= P_a \otimes 1 + e^{-zP_0} \otimes P_a, \\
\Delta_z(J) &= J \otimes 1 + 1 \otimes J, \\
\Delta_z(K_a) &= K_a \otimes 1 + e^{-zP_0} \otimes K_a + z\,\epsilon_{abc}P_b \otimes J_c\,.
\end{aligned}
$$

- Poisson algebra dual to translations

$$
\left[X^0, X^i\right] = -z\,X^i, \qquad \left[X^i, X^j\right] = 0
$$

obtained by dualizing the cocommutator map (its form can be read off from the first-order deformation of the coproducts of P) and gives the k-Minkowski spacetime algebra

this construction is possible because the translations close a Hopf sub-algebra

Preliminaries: revisiting the k-Poincaré momentum space construction

✦ the generic element of the dual Poisson-Lie group is constructed via exponentiation, with coordinates on the group p_μ

$$G^*(p_0, p_1, p_2) = \exp\left(p_1 \rho(X^1)\right) \exp\left(p_2 \rho(X^2)\right) \exp\left(p_0 \rho(X^0)\right),$$

the coproducts of P_μ can be re-obtained from the group law of G* upon identifying $P_\mu \equiv p_\mu$
a different choice of ordering of the exponentials would result in a different choice of basis of the translation generators

✦ 4d representation of X^μ

$$\rho(X^0) = z \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(X^1) = z \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \rho(X^2) = z \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

✦ then the group element reads

$$G^*(p) = \begin{pmatrix} \cosh(zp_0) + \frac{1}{2} e^{z\,p_0} z^2 \bar{p}^2 & zp_1 & zp_2 & \sinh(zp_0) + \frac{1}{2} e^{z\,p_0} z^2 \bar{p}^2 \\ e^{z\,p_0}\, zp_1 & 1 & 0 & e^{z\,p_0}\, zp_1 \\ e^{z\,p_0}\, zp_2 & 0 & 1 & e^{z\,p_0}\, zp_2 \\ \sinh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \bar{p}^2 & -zp_1 & -zp_2 & \cosh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \bar{p}^2 \end{pmatrix}$$

Preliminaries: revisiting the k-Poincaré momentum space construction

✦ the k-Poincaré momentum space is generated by the orbits of the dual Poisson-Lie group G* acting on the ambient Minkowski space that pass through the point (0,0,0,1):

$$G^* \cdot (0,0,0,1)^T = (S_0, S_1, S_2, S_4)^T \, .$$

✦ where we recover the coordinates defined earlier:

$$\begin{aligned} S_0 &= \sinh(z p_0) + \tfrac{1}{2}\, e^{z\,p_0}\, z^2\, \bar{p}^2, \\ S_1 &= e^{z\,p_0}\, z\, p_1, \\ S_2 &= e^{z\,p_0}\, z\, p_2, \\ S_4 &= \cosh(z p_0) - \tfrac{1}{2}\, e^{z\,p_0}\, z^2\, \bar{p}^2 \, . \end{aligned}$$

such that $-S_0^2 + S_1^2 + S_2^2 + S_4^2 = 1$ and $S_0 + S_4 = e^{z\,p_0} > 0$

✦ this defines half of a 2+1 dimensional de Sitter manifold

(note the crucial role of the coproducts in the construction)

• *J. Kowalski-Glikman, Int. J. Mod. Phys. A 28 (2013)*
• *Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)*
• *Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD97 (2018)*

k-(anti) de Sitter algebra

- in the k-(anti) de Sitter algebra we see explicitly at work the nontrivial interplay between the 'quantum' deformation parameter z and the cosmological constant Λ, that is a classical deformation parameters ($\Lambda > 0$ de Sitter, $\Lambda < 0$ anti de Sitter)

- algebra in 2+1 dimensions (bicrossproduct basis)

• *Ballesteros, Herranz, del Olmo, Santander, J. Phys. A (1994)*

$$
\begin{aligned}
&\{J, P_0\} = 0, \qquad \{J, P_1\} = P_2, \qquad \{J, P_2\} = -P_1, \\
&\{J, K_1\} = K_2, \qquad \{J, K_2\} = -K_1, \qquad \{K_1, K_2\} = -\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}}, \\
&\{P_0, P_1\} = -\Lambda\, K_1, \quad \{P_0, P_2\} = -\Lambda\, K_2, \quad \{P_1, P_2\} = \Lambda\, \frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}}, \\
&\{K_1, P_0\} = P_1, \qquad \{K_2, P_0\} = P_2, \\
&\{P_2, K_1\} = z\,(P_1 P_2 - \Lambda K_1 K_2) \qquad \{P_1, K_2\} = z\,(P_1 P_2 - \Lambda K_1 K_2)\,, \\
&\{K_1, P_1\} = \frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J) - e^{-2zP_0}\right) + \frac{z}{2}\,(P_2^2 - P_1^2) - \frac{z\Lambda}{2}\,(K_2^2 - K_1^2)\,, \\
&\{K_2, P_2\} = \frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J) - e^{-2zP_0}\right) + \frac{z}{2}\,(P_1^2 - P_2^2) - \frac{z\Lambda}{2}\,(K_1^2 - K_2^2)\,,
\end{aligned}
$$

- coproducts

$$
\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J) = J \otimes 1 + 1 \otimes J, \\
\Delta(P_1) &= P_1 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes P_1 + \Lambda\, K_2 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(P_2) &= P_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes P_2 - \Lambda\, K_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(K_1) &= K_1 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes K_1 + P_2 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(K_2) &= K_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes K_2 - P_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},
\end{aligned}
$$

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&\{P_0, P_1\} = -\Lambda\, K_1, \quad \{P_0, P_2\} = -\Lambda\, K_2, \quad \{P_1, P_2\} = \Lambda\, \frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},\\
&\{K_1, P_0\} = P_1, \qquad \{K_2, P_0\} = P_2,\\
&\{P_2, K_1\} = z\,(P_1P_2 - \Lambda K_1K_2) \qquad \{P_1, K_2\} = z\,(P_1P_2 - \Lambda K_1K_2)\,,\\
&\{K_1, P_1\} = \frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J) - e^{-2zP_0}\right) + \frac{z}{2}\,(P_2^2 - P_1^2) - \frac{z\Lambda}{2}\,(K_2^2 - K_1^2)\,,\\
&\{K_2, P_2\} = \frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J) - e^{-2zP_0}\right) + \frac{z}{2}\,(P_1^2 - P_2^2) - \frac{z\Lambda}{2}\,(K_1^2 - K_2^2)\,,
\end{aligned}
$$

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\Delta(P_2) &= P_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes P_2 - \Lambda\, K_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},\\
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\Delta(K_2) &= K_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes K_2 - P_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},
\end{aligned}
$$

- spacetime translations do not close a sub algebra any more

k-(anti) de Sitter algebra - dual Lie algebra and construction of the momentum space

✦ because in particular the coalgebra sector of translations does not close, the dual Lie algebra needs to be constructed with respect to the whole set of k-de Sitter generators

$$
\begin{array}{lll}
[X^0, X^1] = -z\, X^1, & [X^0, X^2] = -z\, X^2, & [X^1, X^2] = 0, \\
[X^0, L^1] = -z\, L^1, & [X^0, L^2] = -z\, L^2, & [L^1, L^2] = 0, \\
[R, X^2] = -z\, L^1, & [R, L^1] = z\, \Lambda\, X^2, & [L^1, X^2] = 0, \\
[R, X^1] = z\, L^2, & [R, L^2] = -z\, \Lambda\, X^1, & [L^2, X^1] = 0, \\
[R, X^0] = 0, & [L^1, X^1] = 0, & [L^2, X^2] = 0.
\end{array}
$$

X are dual to translations P, R is dual to the rotation J and L are dual to boosts K

✦ the generic element of the corresponding dual Poisson-Lie group G* is again defined via exponentiation

$$G^*_\Lambda = \exp\left(\theta\rho(R)\right) \exp\left(p_1\rho(X^1)\right) \exp\left(p_2\rho(X^2)\right) \exp\left(\chi_1\rho(L^1)\right) \exp\left(\chi_2\rho(L^2)\right) \exp\left(p_0\rho(X^0)\right)$$

however now the local group coordinates include 'generalized momenta' θ, χ_i, associated to boosts and rotations, besides spacetime translations

as seen before, the coproducts and algebra of the q-de Sitter algebra can be recovered from the group law of G* upon identifying $P_\mu \equiv p_\mu, \quad \chi_i \equiv K_i, \quad \theta = J$

✦ the orbits of the dual Poisson-Lie group G* acting on the ambient Minkowski space that pass through the point (0,0,0,0,0,1) generate the momentum space of k-de Sitter and are given by

$$G^* \cdot (0,0,0,0,0,1)^T = (S_0, S_1, S_2, S_3, S_4, S_5)^T\ .$$

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB 2017*
•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD 2018*

k-de Sitter algebra - properties of the generalised momentum space

- the orbits are parameterised by

$$
\begin{aligned}
S_0 &= \sinh(zp_0) + \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right) \\
S_1 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_1 - \sqrt{\Lambda}\, \sin(z\sqrt{\Lambda}\,\theta)\, \chi_2\right), \\
S_2 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\, \sin(z\sqrt{\Lambda}\,\theta)\, \chi_1\right), \\
S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\, \cos(z\sqrt{\Lambda}\,\theta)\, \chi_1\right), \\
S_4 &= e^{z\,p_0} z \left(\sin(z\sqrt{\Lambda}\,\theta)\, p_1 + \sqrt{\Lambda}\, \cos(z\sqrt{\Lambda}\,\theta)\, \chi_2\right), \\
S_5 &= \cosh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right)
\end{aligned}
$$

these coordinates satisfy the conditions $-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1$ and $S_0 + S_5 = e^{z\,p_0} > 0$

this is the embedding of (half of) a 4+1 dimensional de Sitter space into the ambient 5+1 dimensional Minkowski space, M_{dS_5}

(note symmetric role of spatial momenta and boosts)

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB 2017*
•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD 2018*

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S_1 &= e^{z\,p_0} z\,(\cos(z\sqrt{\Lambda}\,\theta)\,p_1 - \sqrt{\Lambda}\,\sin(z\sqrt{\Lambda}\,\theta)\,\chi_2), \\
S_2 &= e^{z\,p_0} z\,(\cos(z\sqrt{\Lambda}\,\theta)\,p_2 + \sqrt{\Lambda}\,\sin(z\sqrt{\Lambda}\,\theta)\,\chi_1), \\
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S_4 &= e^{z\,p_0} z\,(\sin(z\sqrt{\Lambda}\,\theta)\,p_1 + \sqrt{\Lambda}\,\cos(z\sqrt{\Lambda}\,\theta)\,\chi_2), \\
S_5 &= \cosh(zp_0) - \frac{1}{2}\, e^{z\,p_0}\, z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right)
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these coordinates satisfy the conditions $-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1$ and $S_0 + S_5 = e^{z\,p_0} > 0$

this is the embedding of (half of) a 4+1 dimensional de Sitter space into the ambient 5+1 dimensional Minkowski space, M_{dS_5}

✦ the momentum space has a lower dimensionality compared to the number of generators because rotations generate the isotropy group of the point (0,0,0,0,0,1).
Taking this into account the full momentum space is $M_{dS_5} \times S^1$

•Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB 2017
•Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD 2018

k-de Sitter algebra - properties of the generalised momentum space

- the orbits are parameterised by

$$
\begin{aligned}
S_0 &= \sinh(zp_0) + \frac{1}{2}\, e^{z\,p_0}\, z^2 \left(p_1^2 + p_2^2 + \Lambda \left(\chi_1^2 + \chi_2^2\right)\right) \\
S_1 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_1 - \sqrt{\Lambda}\, \sin(z\sqrt{\Lambda}\,\theta)\, \chi_2\right), \\
S_2 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\, \sin(z\sqrt{\Lambda}\,\theta)\, \chi_1\right), \\
S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\, \cos(z\sqrt{\Lambda}\,\theta)\, \chi_1\right), \\
S_4 &= e^{z\,p_0} z \left(\sin(z\sqrt{\Lambda}\,\theta)\, p_1 + \sqrt{\Lambda}\, \cos(z\sqrt{\Lambda}\,\theta)\, \chi_2\right), \\
S_5 &= \cosh(zp_0) - \frac{1}{2}\, e^{z\,p_0}\, z^2 \left(p_1^2 + p_2^2 + \Lambda \left(\chi_1^2 + \chi_2^2\right)\right)
\end{aligned}
$$

these coordinates satisfy the conditions $-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1$ and $S_0 + S_5 = e^{z\,p_0} > 0$

this is the embedding of (half of) a 4+1 dimensional de Sitter space into the ambient 5+1 dimensional Minkowski space, M_{dS_5}

- the momentum space has a lower dimensionality compared to the number of generators because rotations generate the isotropy group of the point (0,0,0,0,0,1). Taking this into account the full momentum space is $M_{dS_5} \times S^1$

- this allows us to write the dispersion relation associated to the Casimir in a simplified way

$$
\mathcal{C}_z = \frac{2}{z^2}\left[\cosh(zp_0)\cos(z\sqrt{\Lambda}\theta) - 1\right] - e^{zp_0}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)\cos(z\sqrt{\Lambda}\theta) - 2\,\Lambda\, e^{zp_0}\frac{\sin(z\sqrt{\Lambda}\theta)}{\sqrt{\Lambda}} R_3,
$$

$[R_3 = \epsilon_{3bc}\chi_b p_c]$

$$
\theta = 0 \quad \longrightarrow \quad \mathcal{C}_z = \frac{2}{z^2}\left[\cosh(zp_0) - 1\right] - e^{zp_0}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)
$$

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB 2017*

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD 2018*

Conclusions and outlook - 1

✦ Simultaneous presence of curvature in ST and MS intertwines spacetime translations and boosts. Then the momentum space is the manifold is generated by the orbits of the dual Lie group of both translations and boosts. It is a manifold of 'generalised momenta': those associated to translations+ 'hyperbolic momenta' associated to boosts

✦ In 2+1 dimensions, the resulting manifold is half of a de Sitter space with (4+1) dimensions (#of translations+# of boosts). Rotations are the isometry subgroup of the origin of the momentum space.

✦These results can be generalised to q-anti de Sitter algebra, obtaining a manifold defined by the quadratic constraint:

$$-S_0^2+S_1^2+S_2^2-S_3^2-S_4^2+S_5^2=1$$

✦ One can also look at higher dimension - the main nontrivial additional ingredient are rotations, which close a deformed sub algebra with a privileged direction

$$\begin{aligned}\Delta_z(J_3) &= J_3\otimes 1+1\otimes J_3,\\ \Delta_z(J_1) &= J_1\otimes e^{z\sqrt{\omega}J_3}+1\otimes J_1,\\ \Delta_z(J_2) &= J_2\otimes e^{z\sqrt{\omega}J_3}+1\otimes J_2,\end{aligned}$$

still, one can construct the generalised momentum space as half of a 6+1 dimensional de Sitter manifold and the rotation sector only has the role of generating the isotropy subgroup of its origin (0,0,0,0,0,0,0,1)

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PRD 2018*

Conclusions and outlook - 2

✦ Understanding the role of rotations allows us to write the Casimir in a simplified form

$$\mathcal{C}_z = \frac{2}{z^2}\left[\cosh(zp_0) - 1\right] - e^{zp_0}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)$$

which in principle would allow us to study the phenomenology in a similar fashion as the kP case, investigating relative locality effects in a curved spacetime.

✦ This has been done already in the 1+1 case, which however is trivial from the point of view of the momentum space (translations and boosts separate). Still, some interesting features already emerged:

the dispersion relation and worldlines show explicitly the interplay between curvature in ST and MS, already at first order in the deformation parameters:

$$p_0 = -p_1\left(1 - Hx^0 - \ell p_1\left(\frac{1}{2} - Hx^0\right)\right)$$

$$x^1 - \bar{x}^1 = (x^0 - \bar{x}^0)(1 - \ell p_1) - \frac{1}{2}H\left((x^0)^2 - (\bar{x}^0)^2\right)(1 - 2\ell p_1)$$

these features remain in observable properties, such as the time delay in the travel time of photons with different energies and the energy redshift of a photon traveling between far away observers:

$$\Delta x^0 = \ell a^0 \Delta p_0 (1 + Ha^0)$$

$$\Delta p_0 = -Hp_0 x^0 (1 + \frac{\ell}{2} p_0)$$

• *Barcaroli, Gubitosi, PRD 2016*