Inhomogeneous chiral condensates within the Functional Renormalisation Group

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DFG

CRC-TR 211
Strong-interaction matter under extreme conditions

TECHNISCHE UNIVERSITÄT DARMSTADT

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Helmholtz Graduate School for Hadron and Ion Research

Outline

- Motivation and introduction
- Inhomogeneous chiral condensates within the FRG framework
- FRG based mean-field calculations - Part I - *'the ~~naïve~~ wrong way'*
  - Homogeneous and inhomogeneous chiral condensates
- FRG based mean-field calculations - Part II - *'the consistent way'*
  - Consistent parameter fixing
  - Renormalization group consistency
- Summary and outlook

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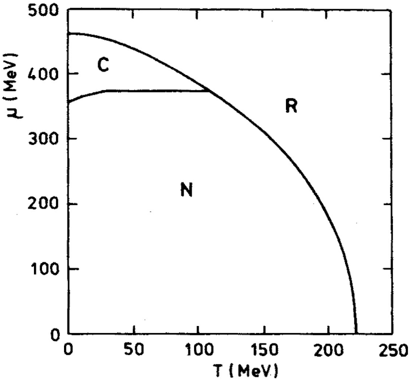
Mean-field phase diagram for the Quark-Meson model (QMM)

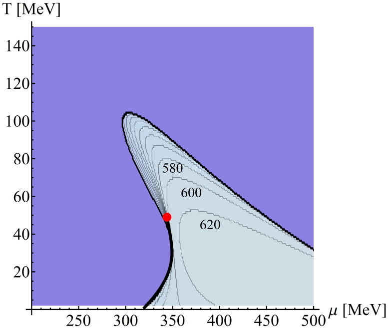


- **N**on-vanishing, homogeneous condensate: $\langle\bar{\psi}\psi\rangle\cancel{(\vec{x})} > 0$
- **R**estored phase with a vanishing homogeneous condensate: $\langle\bar{\psi}\psi\rangle\cancel{(\vec{x})} = 0$
- **C**hiral density wave a non-vanishing, inhomogeneous condensate: $\langle\bar{\psi}\psi\rangle(\vec{x}) > 0$

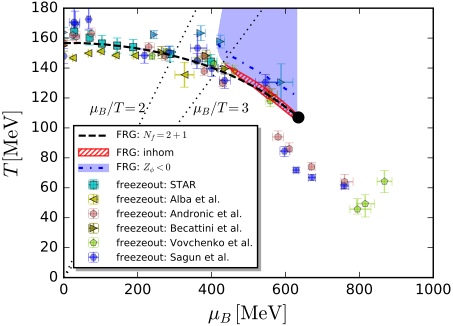
W. Broniowski, A. Kotlorz, M. Kutschera, Acta Phys. Polon. B **22**, 145 (1991)
M. Buballa, S. Carignano, Prog. Part. Nucl. Phys. **81** (2015)

FRG based stability analysis of the homogeneous phase

$N_f = 2$ flavor QMM[1]



$N_f = 2 + 1$ flavor QCD[2]



[1] R.-A. Tripolt, B.-J. Schaefer, L. von Smekal, J. Wambach, Phys. Rev. D97 (2018)
[2] W.-j. Fu, J. M. Pawlowski, F. Rennecke, arXiv: `1909.02991 [hep-th]` (2019)

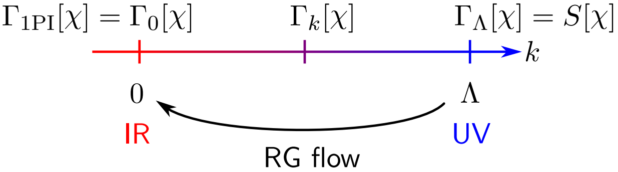
FRG study of inhomogeneous chiral condensates

- **Motivation:** Open questions regarding the stability of inhomogeneous chiral condensates under quantum and thermal fluctuations

- **Current goal:** Study effects of bosonic and fermionic quantum fluctuations on inhomogeneous chiral condensates in the QMM
  - $N_f = 2$ quark-meson model in the chiral limit
  - Chiral density wave (CDW) ansatz for the inhomogeneous chiral condensate

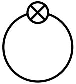
- **Method:** Study within the Functional Renormalization Group (FRG)
  - Highly potent tool to investigate effects of quantum fluctuations
  - In-medium computations ($T \geq 0$ and $\mu \geq 0$) are possible
  - Inclusion of inhomogeneous condensates is formally unproblematic

- Implementation of Wilson's RG approach:



- Exact renormalization group equation

$$\frac{\mathrm{d}\Gamma_k[\chi]}{\mathrm{d}k} = \frac{1}{2}\,\mathrm{STr}\left\{\left[\Gamma_k^{(2)}[\chi] + R_k\right]^{-1} \partial_k R_k\right\} = \frac{1}{2}$$



C. Wetterich Phys. Lett. B **301**.1 (1993); Wilson, Phys. Rev. B **4** 9 (1971)
J. Berges, N. Tetradis, C. Wetterich, Phys. Rept. **363** (2002)

Two flavor Quark-Meson model in LPA with CDW

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- Truncation of Γ_k is necessary to explicitly solve the flow equation: Lowest-order *derivative expansion*: **Local potential approximation (LPA) for QM model** in the chiral limit:

$$\Gamma_{k}[\psi,\bar{\psi},\phi] = \int \mathrm{d}^4 z \Big\{ \bar{\psi}(z) \Big[\partial\!\!\!/ + \gamma_0 \mu + g\big(\sigma(z) + \mathrm{i}\gamma_5 \vec{\tau}\cdot\vec{\pi}(z)\big)\Big] \psi(z) + \\ + \frac{1}{2}\big(\partial_\mu \phi(z)\big)\big(\partial^\mu \phi(z)\big) + U_{k}(\phi(z)^2/2) \Big\}$$

- **Chiral density wave (CDW) ansatz for the condensates:**

$$\phi(z) \overset{\mathrm{CDW}}{=} \big(\sigma(\vec{z}), 0, 0, \pi_3(\vec{z})\big) = \frac{M}{g} \big(\cos(\vec{q}\cdot\vec{z}), 0, 0, \sin(\vec{q}\cdot\vec{z})\big)$$

$$2\rho(z) \equiv \phi(z)\phi(z) \overset{\mathrm{CDW}}{=} \frac{M^2}{g^2}$$ Spatially independent $O(4)$-sym. field

$$\sigma(z) \pm \mathrm{i}O\pi_3(z) \overset{\mathrm{CDW}}{=} \frac{M}{g} \exp\left(\pm \mathrm{i} O\, \vec{q}\cdot\vec{z}\right),$$ for $O^2 = \mathbb{1}$ *Euler's formula*

Two-point functions with CDW condensates

- **Challenge:** Non trivial position dependence for the CDW in

$$\Gamma_k^{(0,1,1)}(x,y) \equiv \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\Gamma_k[\psi,\bar{\psi},\phi]\frac{\overleftarrow{\delta}}{\delta\psi(y)}$$
$$\overset{\text{CDW}}{=} \delta^{(4)}(x-y)\Big[\partial\!\!\!/_x + \gamma_0\mu + M\left(\cos(\vec{q}\cdot\vec{x}) + \mathrm{i}\gamma_5\tau_3\sin(\vec{q}\cdot\vec{x})\right)\Big]$$
$$= \delta^{(4)}(x-y)\Big[\partial\!\!\!/_x + \gamma_0\mu + M\exp\left(\mathrm{i}\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)\Big]$$

$$\Gamma_k^{(2,0,0)}(x,y) \equiv \frac{\delta}{\delta\phi_i(x)}\frac{\delta}{\delta\phi_j(y)}\Gamma_k[\psi,\bar{\psi},\phi]$$
$$\overset{\text{CDW}}{=} \delta^{(4)}(x-y)\Big[\left(-\partial_x^2 + U_k'(\rho)\right)\delta_{ij} + U_k''(\rho)\phi_i(x)\phi_j(x)\Big]$$

- **Solution:** Construct unitary transformation ($U^\dagger U = \mathbb{1}$ and $\partial_k U = 0$) for the CDW analytically to eliminate explicit position dependence $\Leftrightarrow$ diagonalize $\Gamma_k^{(2)}$ in momentum space

Unitary transformations for the CDW

- **The transformation for the fermionic two-point function:**

$$U_F(\vec{x}) \equiv \exp\left(-\frac{\mathrm{i}}{2}\gamma_5\tau_3\vec{q}\cdot\vec{x}\right)$$

diagonalizes $\gamma_0\Gamma_k^{(0,1,1)}$ in momentum space.

- **The transformation for the bosonic two-point function:**

$$U_B(\vec{x}) \equiv \frac{1}{2}\begin{pmatrix} 1-\exp(-2\mathrm{i}\vec{q}\cdot\vec{x}) & 0 & 0 & 1+\exp(-2\mathrm{i}\vec{q}\cdot\vec{x}) \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -\mathrm{i}(1+\exp(-2\mathrm{i}\vec{q}\cdot\vec{x})) & 0 & 0 & \mathrm{i}(\exp(-2\mathrm{i}\vec{q}\cdot\vec{x})-1) \end{pmatrix}.$$

diagonalizes $\Gamma_k^{(2,0,0)}$ in momentum space.

LPA Flow equation

LPA flow equation for $U_k(\rho)$ with CDW condensates

$$\partial_k U_k(\rho) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{i=0}^{3} \frac{1}{2} \coth\left(\frac{E_k^i}{2T}\right) \partial_k E_k^i + \\ - 2N_c \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{\pm,\pm} \tanh\left(\frac{E_k^{\pm} \pm \mu}{2T}\right) \partial_k E_k^{\pm}$$

- Using generic but **three-dimensional** FRG regulators

$$R_k^F(p,p') \equiv -\mathrm{i}\vec{\not{p}}\, r_k^F(|\vec{p}|/k)(2\pi)^4 \delta^{(4)}(p-p')$$
$$R_k^B(p,p') \equiv \vec{p}^{\,2} r_k^B(|\vec{p}|/k)(2\pi)^4 \delta^{(4)}(p-p')$$

in a unified regulator scheme

$$\left(1 + r_k^F(|\vec{p}|/k)\right)^2 = 1 + r_k^B(|\vec{p}|/k) \equiv \left(\lambda_k(|\vec{p}|)\right)^2.$$

Flowing energy eigenvalues of the CDW

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- **Fermionic eigenvalues**

$$(E_k^\pm)^2 = M^2 + \frac{(\vec{p}_k^{\,+q})^2}{2} + \frac{(\vec{p}_k^{\,-q})^2}{2} + \pm\sqrt{M^2\left(\vec{p}_k^{\,+q} - \vec{p}_k^{\,-q}\right)^2 + \frac{1}{4}\left((\vec{p}_k^{\,+q})^2 - (\vec{p}_k^{\,-q})^2\right)^2}$$

$$\stackrel{q=0}{=} M^2 + (\vec{p}_k)^2$$

with $\vec{p}_k^{\,q} \equiv \left(\vec{p} + \vec{q}/2\right)\left(1 + r_k^F(|\vec{p} + \vec{q}/2|/k)\right) = \left(\vec{p} + \vec{q}/2\right)\lambda_k(|\vec{p} + \vec{q}/2|)$

- **Bosonic eigenvalues**

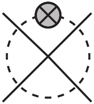
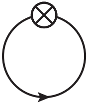
$$(E_k^1)^2 = (E_k^2)^2 = (\vec{p}_k)^2 + U_k'(\rho) \stackrel{q=0}{=} (\vec{p}_k)^2 + U_k'(\rho)$$

$$(E_k^{0,3})^2 = \frac{1}{2}(\vec{p}_k)^2 + \frac{1}{2}(\vec{p}_k^{\,+4q})^2 + U_k'(\rho) + \rho\, U_k''(\rho) + \pm\sqrt{\rho^2 U_k''(\rho)^2 + \frac{1}{4}\left((\vec{p}_k^{\,+4q})^2 - (\vec{p}_k)^2\right)^2}$$

$$\stackrel{q=0}{=} (\vec{p}_k)^2 + U_k'(\rho) + \rho(U_k''(\rho) \pm |U_k''(\rho)|)$$

FRG based mean-field calculations - Part I

- **Mean-field approximation (MFA)** in the present RG setting: Neglect bosonic fluctuations and integrate the LPA flow equation.

$$\partial_k \Gamma_k = \frac{1}{2}$$  $$-$$ 

- UV initial condition

$$U_\Lambda(\rho) = \lambda_\Lambda \rho^2 + m_\Lambda^2 \rho$$

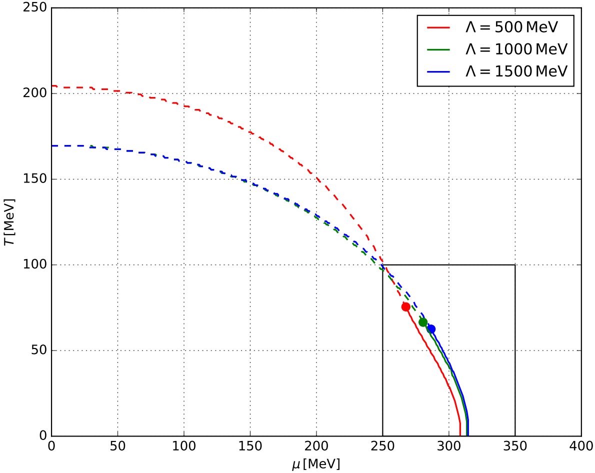
- Exponential regulator shape function

$$\left(1 + r_k^F(|\vec{p}|/k)\right)^2 = \left(\exp(\vec{p}^2/k^2) - 1\right)^{-1} + 1$$

- Model parameters $(g, \lambda_\Lambda, m_\Lambda)$ are fitted by fixing the bare pion decay constant f_π^{b}, the curvature mass of the sigma meson m_σ^{c} and the quark-mass M to 'physical' values

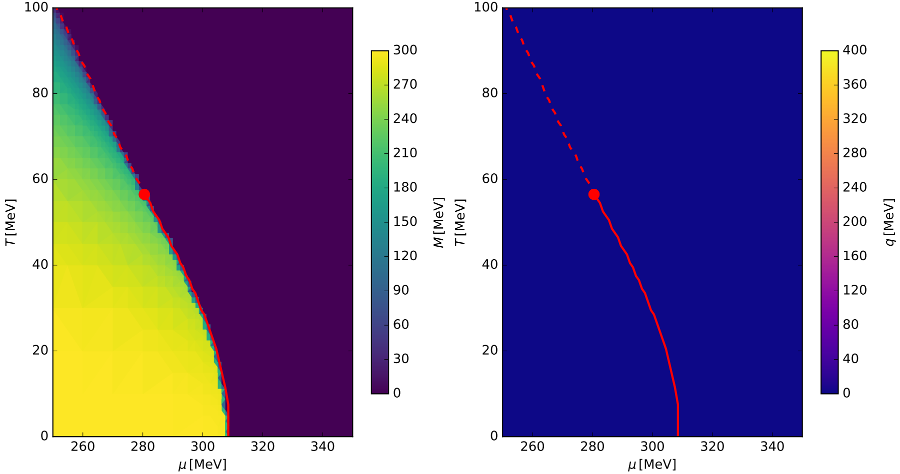
Homogeneous RG MF phase diagrams

$f_\pi^{\mathrm{b}} = 88\,\mathrm{MeV}$, $M = 300\,\mathrm{MeV}$ and $m_\sigma^{\mathrm{c}} = 600\,\mathrm{MeV}$



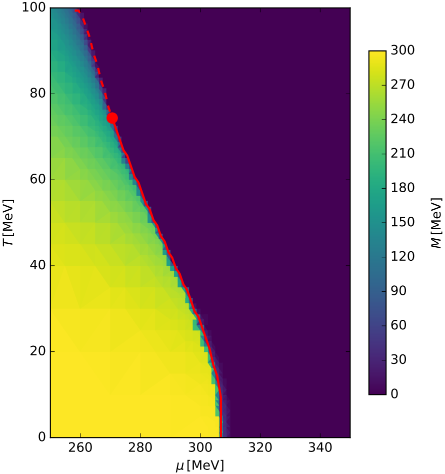
Inhomogeneous RG MF phase diagrams

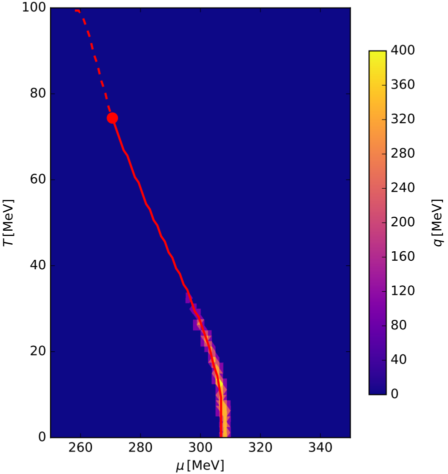
$f_\pi^{\mathrm{b}} = 88\,\mathrm{MeV}$, $M = 300\,\mathrm{MeV}$ and $m_\sigma^{\mathrm{c}} = 600\,\mathrm{MeV}$, $\Lambda = 500\,\mathrm{MeV}$



Inhomogeneous RG MF phase diagrams

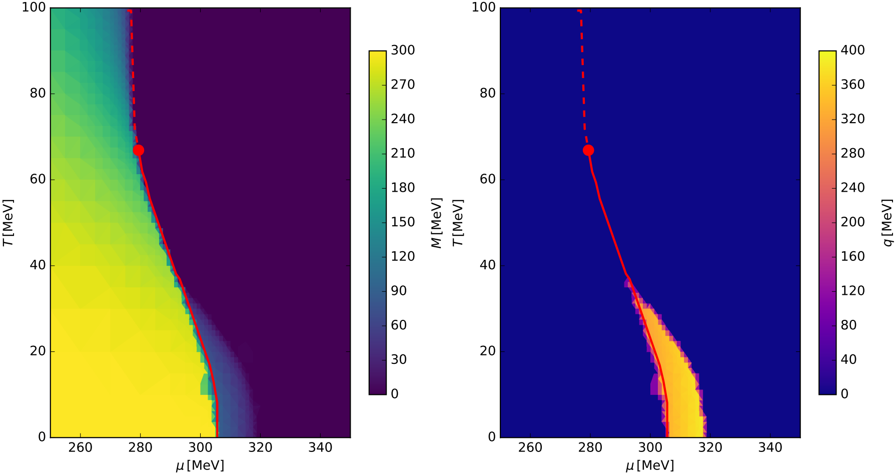
$f_\pi^{\mathrm{b}} = 88\,\mathrm{MeV}$, $M = 300\,\mathrm{MeV}$ and $m_\sigma^{\mathrm{c}} = 600\,\mathrm{MeV}$, $\Lambda = 450\,\mathrm{MeV}$





Inhomogeneous RG MF phase diagrams

$f_\pi^{\mathrm{b}} = 88\,\mathrm{MeV}$, $M = 300\,\mathrm{MeV}$ and $m_\sigma^{\mathrm{c}} = 600\,\mathrm{MeV}$, $\Lambda = 400\,\mathrm{MeV}$

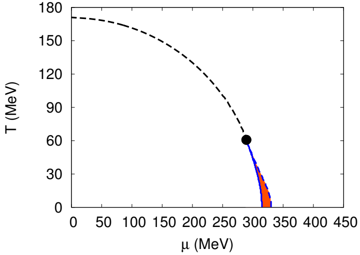


Comparison to existing inhom. mean-field results

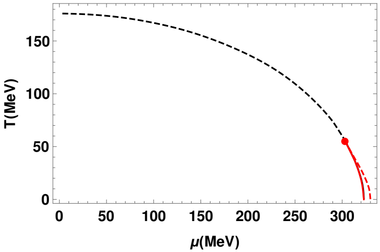
- Involved existing MF results (with $M = 300\,\mathrm{MeV}$, $m_\sigma = 2M$)
  - PV regularization and 'RP' parameter fixing at $\Lambda_{\mathrm{PV}} = 5.0\,\mathrm{GeV}$[1]
  - Dim. regularization using the on-shell (OS) renormalization scheme[2]

are in agreement and predicts a **non-vanishing inhomogeneous window**:

PV 'RP': $f_\pi = 88\,\mathrm{MeV}$



dim. reg. 'OS': $f_\pi = 93\,\mathrm{MeV}$



[1] S. Carignano, M. Buballa, W. Elkamhawy, Phys. Rev. D **94** 3 (2016)
[2] P. Adhikari, J. O. Andersen, P. Kneschke, Phys. Rev. D **96** 1 (2017)

Improving on the naïve RG MFA: RG MF - Part II

- Improved/consistent parameter fixing using $\Gamma^{(2)}_{k=0}$ in MF
  - Fitting renormalized pion decay constant f^{r}_{π} (not f^{b}_{π})
  - Fitting pole-mass m^{p}_{σ} (not m^{c}_{σ})
  - Motivated by MF studies with Pauli-Villars regularization[1]

- *RG-consistent* MF[2] by enforcing:

$$\Lambda \frac{\mathrm{d}\Gamma_{k=0}}{\mathrm{d}\Lambda} = 0$$

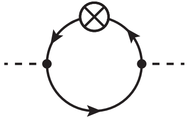
  - Initial condition $\Gamma_{\Lambda'}[\rho]$ at $\Lambda' < \Lambda$ and construction of $\Gamma_{\Lambda}[\rho]$ via RG-consistency
  - Allows for systematic study of cutoff effects and regularization-scheme dependence

[1] S. Carignano, M. Buballa, W. Elkamhawy, Phys. Rev. D **94** 3 (2016)
[2] J. Braun, M. Leonhardt, J. M. Pawlowski, SciPost Phys. **6** (2019)

Mesonic two-point function in RG MF

- Evaluating the flow eq. of the bosonic two-point function on the MF RG flow at $T = \mu = 0$ and at the physical minimum yields:

$$\frac{\mathrm{d}}{\mathrm{d}k}\,\Gamma_k^{\phi\phi}(p_{\mathrm{I}}^0, \vec{p}_{\mathrm{I}}) = 2\,G_{k;\psi\bar{\psi}}\Gamma^{\bar{\psi}\phi\psi}G_{k;\psi\bar{\psi}}\Gamma^{\bar{\psi}\phi\psi}G_{k;\psi\bar{\psi}}\partial_k R_k^{\bar{\psi}\psi}$$

$= 2$ 

- Retarded 2-point function:

$$\Gamma_\phi^{(2),R}(\omega,\vec{p}) = \lim_{\epsilon\to 0}\Gamma_0^{\phi\phi}(p_{\mathrm{I}}^0 = -\mathrm{i}(\omega + \mathrm{i}\epsilon), \vec{p})$$
$$= -Z_{\phi;\Lambda}^{\parallel}\omega^2 + Z_{\phi;\Lambda}^{\perp}\vec{p}^{\,2} + 2\lambda_\Lambda(1 + 2\delta_{\phi\sigma})\rho + m_\Lambda^2 + L_\phi^\Lambda(\omega,\vec{p})$$

with $Z_{\phi;\Lambda}^{\perp} = 1$ and $Z_{\phi;\Lambda}^{\parallel}$ choosen to realise $Z_{\phi;0}^{\parallel} = Z_{\phi;0}^{\perp}$ in the IR

Consistent (RP) parameter fixing

- Consistent scheme: including vacuum fermionic fluctuations by fitting the **r**enormalized pion-decay constant f_π^{r}, the sigma **p**ole mass m_σ^{p} and the quark mass M to 'physical' values

  - We define the sigma pole mass m_σ^{p} as

$$\begin{aligned} 0 &= \mathrm{Re}\,\Gamma_\sigma^{(2),R}(m_\sigma^{\mathrm{p}}, \vec{0}) \\ &= -Z_{\sigma;\Lambda}^{\parallel}(m_\sigma^{\mathrm{p}})^2 + 6\lambda_\Lambda\rho + m_\Lambda^2 + \mathrm{Re}\,L_\sigma^\Lambda(m_\sigma^{\mathrm{p}}, \vec{0}). \end{aligned}$$

  - For the renormalized pion-decay constant

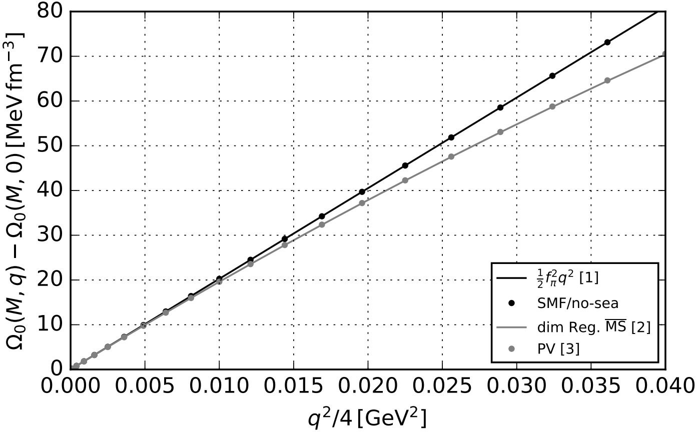
$$f_\pi^{\mathrm{r}} = \left(Z_{\phi;\Lambda}^{\perp}\right)^{1/2} f_\pi^{\mathrm{b}}$$

  we extract the wave function renormalization from

$$Z_{\phi;0}^{\perp} = \frac{1}{2}\left(\frac{\partial^2}{\partial \vec{p}^2}\mathrm{Re}\,\Gamma_\phi^{(2),R}(\omega, \vec{p})\right)_{\omega=0,\,\vec{p}=0}$$

CDW vs. homogeneous ground state at $T = \mu = 0$

Existing MF results



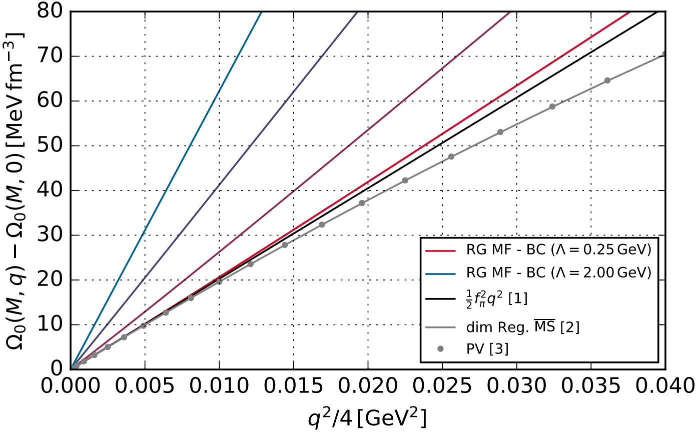
[1] W. Broniowski, M. Kutschera, Phys.Lett. B242, 133 (1990)

[2] P. Adhikari, J. O. Andersen, P. Kneschke, Phys. Rev. D 96, 016013 (2017)

[3] S. Carignano, M. Buballa, B.-J. Schaefer Phys. Rev. D 90, 014033 (2014)

CDW vs. homogeneous ground state at $T = \mu = 0$

RG MF using naïve BC parameter fitting



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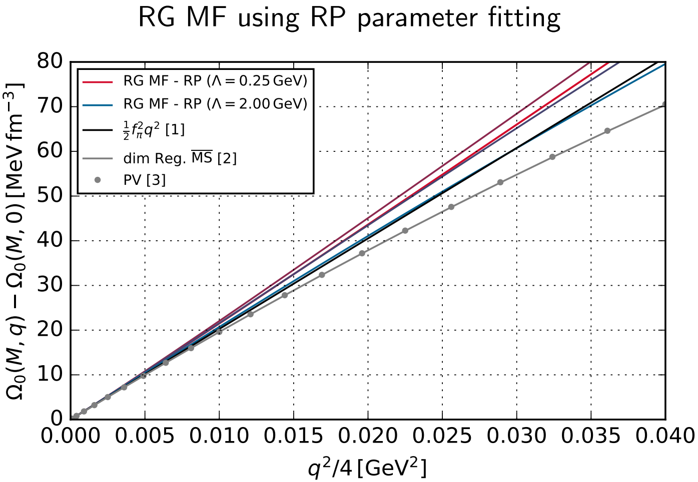
CDW vs. homogeneous ground state at $T = \mu = 0$



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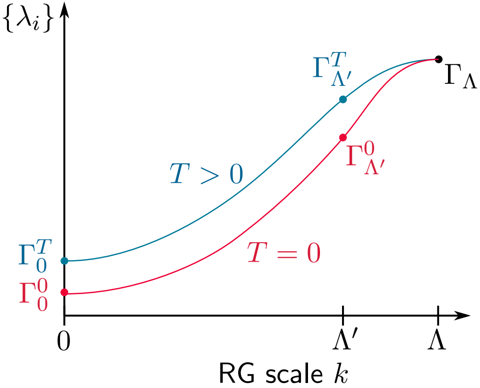
[2] P. Adhikari, J. O. Andersen, P. Kneschke, Phys. Rev. D 96, 016013 (2017)

[3] S. Carignano, M. Buballa, B.-J. Schaefer Phys. Rev. D 90, 014033 (2014)

RG consistency at finite T (and μ)

- RG consistent construction of Γ_Λ to ensure

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\Lambda\frac{\mathrm{d}\Gamma_\Lambda}{\mathrm{d}\Lambda}\right) = 0 \quad \Rightarrow \quad \Gamma_{\Lambda'}(T) \text{ for } \Lambda' < \Lambda$$

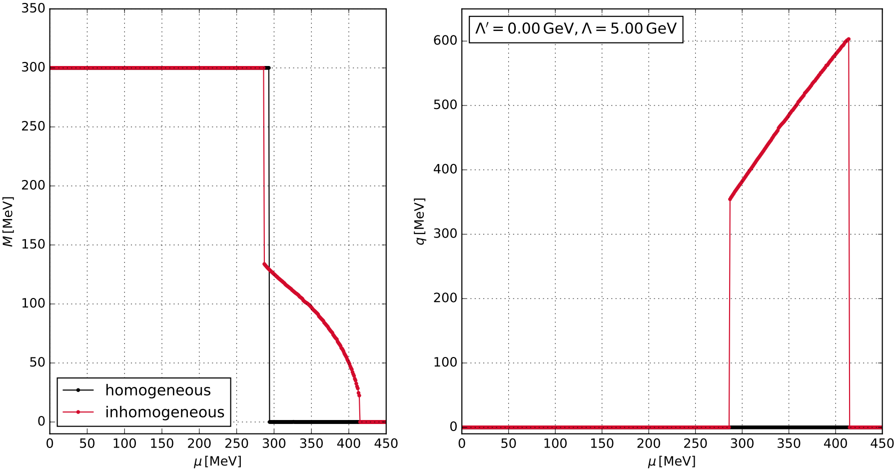


J. Braun, M. Leonhardt, J. M. Pawlowski, SciPost Phys. **6** (2019)

Inhomogeneous window at zero temperature

RG consistent MF: $\Lambda' = 0.00\,\mathrm{GeV}$, $\Lambda = 5.00\,\mathrm{GeV}$ (no-sea)

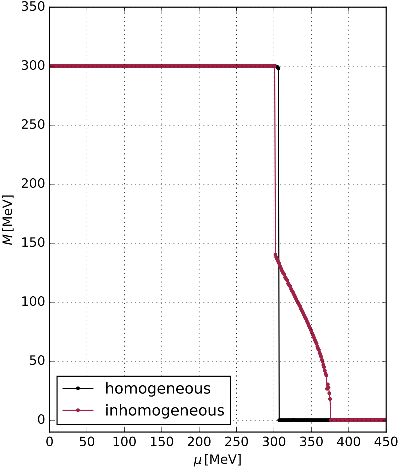
PV parameter fitting: $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$

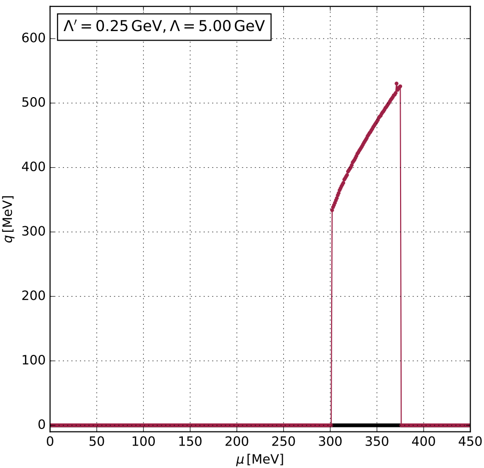


Inhomogeneous window at zero temperature

RG consistent MF: $\Lambda' = 0.25\,\mathrm{GeV}$, $\Lambda = 5.00\,\mathrm{GeV}$

PV parameter fitting: $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$

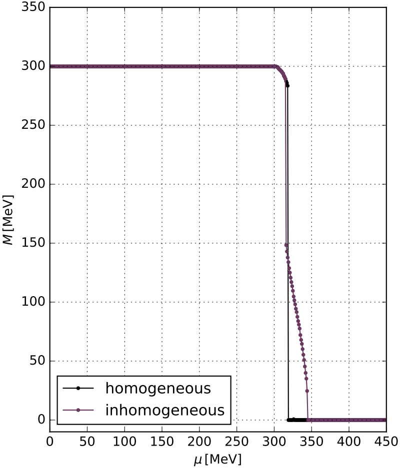


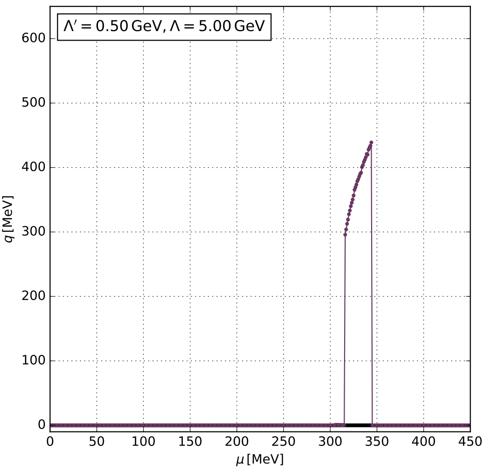


Inhomogeneous window at zero temperature

RG consistent MF: $\Lambda' = 0.50\,\mathrm{GeV}$, $\Lambda = 5.00\,\mathrm{GeV}$

PV parameter fitting: $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$

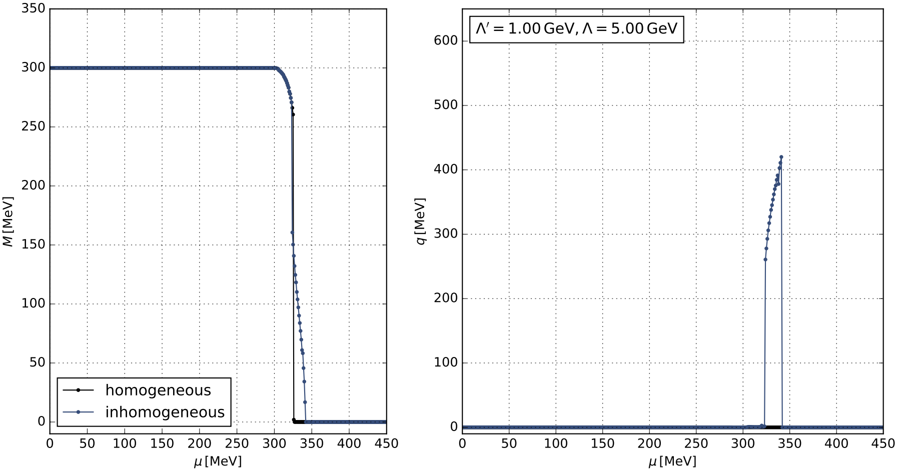




Inhomogeneous window at zero temperature

RG consistent MF: $\Lambda' = 1.00\,\mathrm{GeV}$, $\Lambda = 5.00\,\mathrm{GeV}$

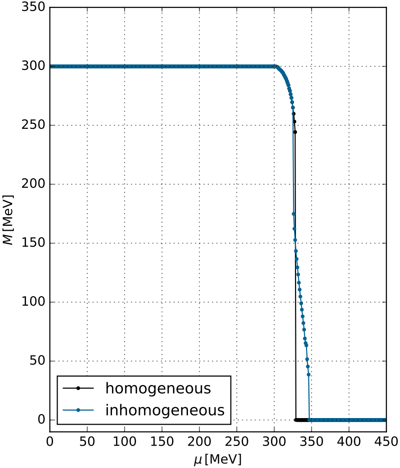
PV parameter fitting: $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$

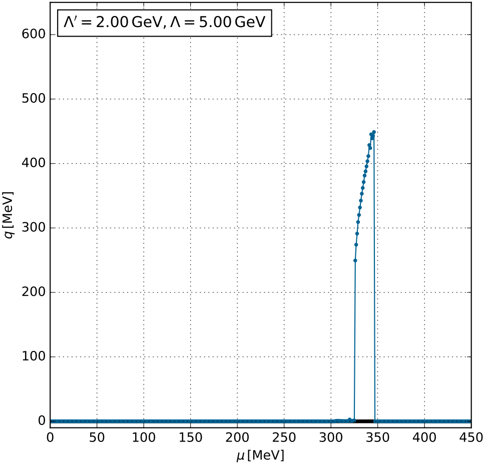


Inhomogeneous window at zero temperature

RG consistent MF: $\Lambda' = 2.00\,\mathrm{GeV}$, $\Lambda = 5.00\,\mathrm{GeV}$

PV parameter fitting: $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$





Summary and outlook

- **What we have done so far:**
  - Derivation of a LPA flow eq. for inhomogeneous CDW condensates
  - Numerical results of FRG based mean-field computations
    - RG consistency, fermionic contributions to $\Gamma_k^{\phi\phi}(p_{\mathrm{I}}^0, \vec{p}_{\mathrm{I}}) \Rightarrow f_\pi^{\mathrm{r}}, m_\sigma^{\mathrm{p}}$
    - Qualitative agreement with existing MF results
    - Small quantitative deviation from existing MF results: enhanced sensibility on $m_\sigma^{\mathrm{p}} \leftarrow$ regulator choice and Poincaré-invariance
- **What we are currently working on:**
  - Numerical solution of the full CDW flow equation for the CDW
    - finite volume methods for discretization in ρ-direction on a q-grid
    - advanced implicit ODE-time steppers for integration in k
- **What we plan to do in the future:**
  - RG consistent MF study using four-dimensional regulators
  - Systematic comparison to FRG based stability analysis of the homogeneous phase
  - Extending the truncation: deriving flow equations beyond LPA in presence of CDW condensates

Questions and discussion

xkcd.com/1739/