UNIFIED APPROACH TO SPIN HYDRODYNAMICS

Wojciech Florkowski

Institute of Theoretical Physics, Jagiellonian University, Kraków, Poland

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based on: **WF + M. Hontarenko**, arXiv:2405.03263

Outline

1 Introduction

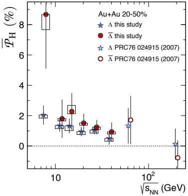
1.1 Is QGP the most vortical fluid?

First positive measurements of Λ spin polarization

Non-central heavy-ion collisions create fireballs with large global angular momenta which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions, both from the experimental and theoretical point of view
L. Adamczyk et al. (**STAR**), (2017), **Nature 548 (2017) 62-65**, arXiv:1701.06657 (nucl-ex)

Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever

The $\sqrt{s_{NN}}$-averaged polarizations indicate a vorticity of $\omega \approx (9 \pm 1) \times 10^{21}\,\mathrm{s}^{-1}$, with a systematic uncertainty of a factor of two, mostly owing to uncertainties in the temperatur This far surpasses the vorticity of all other known fluids, including solar subsurface flow[23] ($10^{-7}\,\mathrm{s}^{-1}$); large-scale terrestrial atmospheric patterns[24] (10^{-7}–$10^{-5}\,\mathrm{s}^{-1}$); supercell tornado cores[25] ($10^{-1}\,\mathrm{s}^{-1}$); the great red spot of Jupiter[26] (up to $10^{-4}\,\mathrm{s}^{-1}$); and the rotating, heated soap bubbles ($100\,\mathrm{s}^{-1}$) used to model climate change[27]. Vorticities of up to $150\,\mathrm{s}^{-1}$ have been measured in turbulent flow[28] in bulk superfluid He II, and Gomez *et al.*[29] have recentl produced superfluid nanodroplets with $\omega \approx 10^{7}\,\mathrm{s}^{-1}$.



$\Delta t = 1$ fm/c = 3×10^{-24} s, $\qquad \Delta t\, \omega_{\mathrm{max}} = 27 \times 10^{-24} \times 10^{21} = 2.7 \times 10^{-2}$

Large angular momentum does not mean large rotation!

1.2 Equilibrated spin

J. Weyssenhoff and A. Raabe, Acta Phys. Pol. 9 (1947) 7

revival of interest in hydrodynamics of spin polarized systems
in a series of works by Francesco Becattini and collaborators

Connection betweeen theory and experiment by the "spin Cooper-Frye formula" Pauli-Lubański vector defined by the spin polarization tensor $\omega^{\mu\nu}$

$$\pi^\mu(p) = -\frac{1}{8m}\varepsilon^{\mu\rho\sigma\tau} p_\tau \frac{\int d\Sigma_\lambda p^\lambda n(1-n)\omega_{\rho\sigma}}{\int d\Sigma_\lambda p^\lambda n} \qquad \text{integral over freeze-out hypersurface } \Sigma \tag{1}$$

- spin degrees of freedom are equilibrated, spin-orbit coupling interaction included, asymmetric energy-momentum tensor, the spin polarization tensor is equal to thermal vorticity
  $\omega_{\mu\nu} = \varpi_{\mu\nu} = -1/2\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) \qquad \beta^\mu = u^\mu/T, \quad \beta = 1/T$
  the spin polarization tensor is not an independent hydrodynamic variable
- standard (dissipative) hydro is used, $\omega_{\mu\nu}$ determined by the standard hydrodynamic variables such as T and u^μ
- recent works on extension to include effects of the shear stress tensor
- equilibrium distribution functions obtained from QFT (Dirac field under rotation and acceleration, very specific boundary conditions)
- **great success in decribing global polarization,** problems to explain the longitudinal polarization

1.3 Different formulations of spin hydrodynamics

the case of massive spin-1/2 particles is considered only in this talk

1. THV (thermal vorticity oriented) approach described above

   F. Becattini, L. Tinti, V. Chandra, I. Del Zanna, E. Grossi, M. Buzzegoli, G. Inghirami, I. Karpenko, ...

2. KT (kinetic theory) approach (Wigner functions, classical treatment of spin)

   2.1 LKT - local collisions, spin part of total angular momentum conserved

   B. Friman, WF, A. Jaiswal, E. Speranza, R. Ryblewski, A. Kumar, S. Bhadury, R. Singh, ...

   2.2 NLKT - non-local collisions included, only total angular momentum conserved

   N. Weickgenannt, E. Speranza, D. Rischke, D. Wagner, X.-L. Sheng, ...

3. IS (Israel-Stewart) approach

   K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, K. Fukushima, S. Pu, A. Daher, A. Das, R. Biswas, G. Sarwar, M. Hasanujjaman, J. R. Bhatt, H. Mishra, J.-e. Alam, D.-L. Wang, S. Fang, ...

4. Lagrangian formulation

   G. Torrieri, D. Montenegro, K. J. Goncalves, ...

STATISTICAL PHYSICS VS. THERMODYNAMICS
KINETIC THEORY VS. HYDRODYNAMICS

2 Kinetic-theory → perfect spin hydrodynamics

2.1 Thermodynamic identities

Standard form of thermodynamic relations

extensivity → intensivity rule

$$E + PV = TS + \mu N \quad \rightarrow \quad \varepsilon + P = T\sigma + \mu n \tag{2}$$

first law of thermodynamics and Gibbs-Duhem relation

$$d\varepsilon = Td\sigma + \mu dn, \qquad dP = \sigma dT + nd\mu \tag{3}$$

ε, P, T, σ, μ and n are the local energy density, pressure, temperature, entropy density, baryon chemical potential, and baryon number density

Tensor form of thermodynamic relations (Israel-Stewart)

entropy current

$$\boxed{S^{\mu}_{\rm eq} = \sigma u^{\mu} = P\beta^{\mu} - \xi N^{\mu}_{\rm eq} + \beta_{\lambda} T^{\lambda\mu}_{\rm eq}} \tag{4}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_{\lambda} dT^{\lambda\mu}_{\rm eq}, \qquad d(P\beta^{\mu}) = N^{\mu}_{\rm eq} d\xi - T^{\lambda\mu}_{\rm eq} d\beta_{\lambda} \tag{5}$$

standard notation: $\beta^{\mu} = u^{\mu}/T$, $\beta = \sqrt{\beta^{\lambda}\beta_{\lambda}} = 1/T$, and $\xi = \mu/T$

perfect-fluid forms: $N^{\mu}_{\rm eq} = nu^{\mu}$ and $T^{\lambda\mu}_{\rm eq} = (\varepsilon + P)u^{\lambda}u^{\mu} - Pg^{\lambda\mu} = \varepsilon u^{\lambda}u^{\mu} - P\Delta^{\lambda\mu}$

inclusion of spin, $\Omega_{\alpha\beta}$ - spin chemical potential, $S^{\alpha\beta}$ - spin density tensor

$$\varepsilon + P = T\sigma + \mu n + \tfrac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta} \tag{6}$$

$$d\varepsilon = Td\sigma + \mu dn + \tfrac{1}{2}\Omega_{\alpha\beta}dS^{\alpha\beta}., \qquad dP = \sigma dT + nd\mu + \tfrac{1}{2}S^{\alpha\beta}d\Omega_{\alpha\beta} \tag{7}$$

multiplication of the above equations by the hydrodynamic flow vector u gives
the tensor (Israel-Stewart) form

$$\boxed{S^{\mu}_{\rm eq} = P\beta^{\mu} - \xi N^{\mu}_{\rm eq} + \beta_{\lambda}T^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq}} \tag{8}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_{\lambda}dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \quad d(P\beta^{\mu}) = N^{\mu}_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_{\lambda} + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \tag{9}$$

spin tensor

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} \tag{10}$$

analog to the perfect-fluid forms of $N^{\mu}_{\rm eq}$ and $T^{\lambda\mu}_{\rm eq}$

2.2 Kinetic theory for particles with spin

Internal angular momentum (Mathisson), classical spin vector, extended phase-space

Review: WF, A. Kumar, R. Ryblewski, Prog.Part.Nucl.Phys. 108 (2019) 103709

$$s^{\alpha\beta} = \frac{1}{m}\varepsilon^{\alpha\beta\gamma\delta}p_\gamma s_\delta, \quad p_\alpha s^\alpha = 0, \quad s^\alpha = \frac{1}{2m}\varepsilon^{\alpha\beta\gamma\delta}p_\beta s_{\gamma\delta} \tag{11}$$

$$f^\pm_{\rm eq}(x,p,s) = \exp\left(-p\cdot\beta(x) \pm \xi(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right) \tag{12}$$

macroscopic quantities expressed as the moments of the distribution function

$$N^\mu_{\rm eq} = \int dP\, dS\, p^\mu \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right], \qquad \boxed{\partial_\mu N^\mu_{\rm eq} = 0} \tag{13}$$

$$T^{\mu\nu}_{\rm eq} = \int dP\, dS\, p^\mu p^\nu \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right], \qquad \boxed{\partial_\mu T^{\mu\nu}_{\rm eq} = 0} \tag{14}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = \int dP\, dS\, p^\lambda s^{\mu\nu} \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right], \qquad \boxed{\partial_\lambda S^{\lambda,\mu\nu}_{\rm eq} = 0} \tag{15}$$

conservation laws determine: $\beta(x)$, $\xi(x)$, and $\omega_{\alpha\beta}(x)$ — perfect spin hydrodynamics parametrization of the spin polarization tensor in terms of electric- and magnetic-like components (k and ω) in a direct analogy to MHD

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + t_{\alpha\beta}, \qquad t_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta \tag{16}$$

k and ω are orthogonal to the flow vector: $k \cdot u = 0$ and $\omega \cdot u = 0$

spin tensor in the lowest order of k and ω (linear contribution)

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm eq} &= u^\lambda \left[A(T,\mu)\left(k^\mu u^\nu - k^\nu u^\mu\right) + A_1(T,\mu) t^{\mu\nu} \right] \\ &\quad + \frac{A_3(T,\mu)}{2} \left(t^{\lambda\mu} u^\nu - t^{\lambda\nu} u^\mu + \Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu \right) \\ &= u^\lambda S^{\mu\nu}_{\rm eq} + \text{problem} \end{aligned} \tag{17}$$

problem = term that is not proportional to u^λ
kinetic theory does not lead to the form $S^{\mu,\alpha\beta}_{\rm eq} = u^\mu S^{\alpha\beta}_{\rm eq}$, even in local equilibrium state

different ways to proceed:

- IS approach, the kinetic theory result is ignored, one uses the form $S^{\mu,\alpha\beta}_{\rm eq} = u^\mu S^{\alpha\beta}_{\rm eq}$, even with a further simplifying assumption that $S^{\alpha\beta}_{\rm eq} = A(T,\mu)\omega^{\alpha\beta}$ [phenomenological formula for the spin tensor]

  the phenomenological form is not connected to other versions by a pseudogauge transformation; dissipative spin hydrodynamics including the phenomenological spin tensor as the leading contribution is unstable, even in the second-order theory; inconsistent treatment of thermodynamic relations, terms like $\omega_{\mu\nu} S^{\mu\nu}_{\rm eq}$ treated as first-order corrections although they are of the second order

- LKT approach, only the linear terms are kept

  the resulting formalism describes spin evolving in an external standard hydrodynamic background, since corrections to the energy-momentum tensor and baryon current start with the quadratic terms; moderately unstable solutions found for this scheme, thermodynamic relations become trivial, reduce to the standard ones

a solution stands behind the corner ....

From (11)–(14) one obtains **tensor forms of thermodynamic relations** valid for any value of the spin polarization tensor ω

$$S^\mu_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu, \qquad \mathcal{N}^\mu = \coth\xi\ N^\mu_{\rm eq} \neq P u^\mu \tag{18}$$

$$dS^\mu_{\rm eq} = -\xi dN^\mu_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq} \qquad \text{first law of thermodynamics} \tag{19}$$

$$d\mathcal{N}^\mu = N^\mu_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_\lambda + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \qquad \text{Gibbs-Duhem relations} \tag{20}$$

Expansion up to second order in ω

$$N^\mu_{\rm eq} = \bar{n}(T,\xi,k^2,\omega^2)u^\mu + n_t(T,\xi)t^\mu, \tag{21}$$

$$\begin{aligned} T^{\mu\nu}_{\rm eq} &= \bar{\varepsilon}(T,\xi,k^2,\omega^2)u^\mu u^\nu - \bar{P}_{k\omega}(T,\xi,k^2,\omega^2)\Delta^{\mu\nu} \\ &+P_{k\omega}(T,\xi)(\,k^{\langle\mu}k^{\nu\rangle} + \,\omega^{\langle\mu}\omega^{\nu\rangle}) + P_t(T,\xi)\,(t^\mu u^\nu + t^\nu u^\mu). \end{aligned} \tag{22}$$

$t^\mu = t^{\mu\nu}k_\nu = \epsilon^{\mu\nu\alpha\beta}k_\nu u_\alpha\omega_\beta$

$$T^{\mu\nu}_{\rm eq} = \bar{\varepsilon}(T,\xi,k^2,\omega^2)u^\mu u^\nu - \bar{P}_{k\omega}(T,\xi,k^2,\omega^2)\Delta^{\mu\nu} \tag{23}$$
$$+P_{k\omega}(T,\xi)(\,k^{\langle\mu}k^{\nu\rangle} + \,\omega^{\langle\mu}\omega^{\nu\rangle}) + P_t(T,\xi)\,(t^\mu u^\nu + t^\nu u^\mu).$$

Strong **similarity to the case of anisotropic relativistic magnetohydrodynamics** $E^\mu = 0, B^\mu \neq 0$ (similarly to $k^\mu = 0, t^\mu = 0, \omega^\mu \neq 0$)

$$T^{\mu\nu}_{;\nu} = \left[\left(\epsilon + p_\perp + \frac{B^2}{4\pi}\right)U^\mu U^\nu - \left(p_\perp + \frac{B^2}{8\pi}\right)g^{\mu\nu} + \left(p_\parallel - p_\perp - \frac{B^2}{4\pi}\right)n^\mu n^\nu\right]_{;\nu} = 0, \quad (39)$$

M. Gedalin and I. Oiberman, Phys. Rev. E51 (1994) 4901

3 Positive entropy production → dissipative spin hydrodynamics

3.1 Non-equilibrium entropy current

IS method - replacement of the equilibrium currents by the general ones (equilibrium + non-equilibrium corrections)

$$\boxed{S^\mu = T^{\mu\alpha}\beta_\alpha - \tfrac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^\mu + N^\mu_{\rm eq}} \tag{24}$$

Conservations laws, now for total angular momentum $J = L + S$

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta} \tag{25}$$

entropy production

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \xi + \delta T_s^{\mu\lambda}\partial_\mu\beta_\lambda + \delta T_a^{\mu\lambda}\left(\partial_\mu\beta_\lambda - \omega_{\lambda\mu}\right) - \frac{1}{2}\delta S^{\mu,\alpha\beta}\partial_\mu\omega_{\alpha\beta} \geq 0 \tag{26}$$

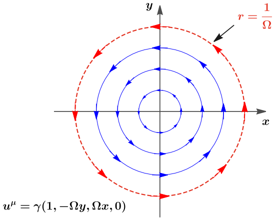
Generalized Tolman-Klein conditions define global equilibrium state:

$$\partial_\mu \xi = 0, \qquad \partial_{(\mu}\beta_{\lambda)} = 0, \qquad \omega_{\lambda\mu} = \partial_{[\mu}\beta_{\lambda]} = -\frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \tag{27}$$

the last condition says that the spin polarization tensor is equal to the thermal vorticity tensor (starting point for F. Becattini's approach)
The middle equation, $\partial_\lambda\beta_\mu + \partial_\mu\beta_\lambda = 0$, is **the Killing equation** with a solution of the form

$$\beta^\mu = \beta^\mu_0 + \varpi^{\mu\nu} x_\nu, \qquad \varpi^{\mu\nu} = -\varpi^{\nu\mu} = \text{const}, \qquad \beta^\mu_0 = \text{const} \tag{28}$$

One possible solution: rigid rotation with a very special boundary condition at $R = 1/\Omega$



3.2 Tensor decompositions

General (mathematical) decomposition of tensors into parts that are: i) either symmetric or antisymmetric, ii) either parallel or orthogonal to the flow, iii) with zero or non-zero trace. The simplest case, baryon current $N^\mu = N^\alpha g^\mu{}_\alpha = N^\alpha(\Delta^\mu{}_\alpha + u^\mu u_\alpha)$

Baryon current

$$N^\mu = a u^\mu + b^\mu \tag{29}$$

here $b^\mu u_\mu = 0$, altogether 4 parameters

Energy-momentum tensor

$$T^{\mu\nu} = c u^\mu u^\nu - e\Delta^{\mu\nu} + d_s^\mu u^\nu + d_s^\nu u^\mu + e_s^{\langle\mu\nu\rangle} + d_a^\mu u^\nu - d_a^\nu u^\mu + e_a^{\mu\nu} \tag{30}$$

here: $d_s^\mu u_\mu = d_a^\mu u_\mu = e_a^{\mu\nu} u_\mu = e_s^{\mu\nu} u_\mu = 0$, $e_s^{\langle\mu\nu\rangle}$ is symmetric and traceless, $e_a^{\mu\nu}$ is antisymmetric
19 parameters, 3 can be eliminated by a suitable choice of the hydrodynamic flow (Landau vs. Eckart), 16 parameters left

Spin tensor

$$S^{\lambda,\mu\nu} = -S^{\lambda,\nu\mu} = u^\lambda \left[(f^\mu u^\nu - f^\nu u^\mu) + \epsilon^{\mu\nu\rho\sigma} u_\rho w_\sigma \right] + i^{\lambda\mu} u^\nu - i^{\lambda\nu} u^\mu + j^{\lambda\mu\nu} \tag{31}$$

here: $f^\mu u_\mu = w^\mu u_\mu = 0$, $i^{\lambda\mu} u_\lambda = i^{\lambda\mu} u_\mu = 0$, $j^{\lambda\mu\nu} = -j^{\lambda\nu\mu}$, $j^{\lambda\mu\nu} u_\lambda = j^{\lambda\mu\nu} u_\mu = j^{\lambda\mu\nu} u_\nu = 0$,
$i^{\lambda\mu}$ can be further decomposed into symmetric (with zero and non-zero trace) and antisymmetric parts
24 parameters

Landau matching conditions

$$N^\mu u_\mu = N^\mu_{\rm eq} u_\mu, \tag{32}$$

$$T^{\mu\nu} u_\mu u_\nu = T^{\mu\nu}_{\rm eq} u_\mu u_\nu, \tag{33}$$

$$S^{\lambda,\mu\nu} u_\lambda = S^{\lambda,\mu\nu}_{\rm eq} u_\lambda. \tag{34}$$

The consequences of Eqs. (32)–(34) are straightforward:

$$a = \bar{n}(T, \xi, k^2, \omega^2), \tag{35}$$

$$c = \bar{\varepsilon}(T, \xi, k^2, \omega^2), \tag{36}$$

$$f^\mu = A(T, \xi) k^\mu, \tag{37}$$

$$w^\mu = A_1(T, \xi) \omega^\mu. \tag{38}$$

one can choose T, ξ, k^μ, and ω^μ (by solving Eqs. (35)–(38)) in such a way that certain parts of N^μ, $T^{\mu\alpha}$, and $S^{\mu,\alpha\beta}$ have the form of the equilibrium tensor

3.3 Navier-Stokes limit

components of the total baryon current

$$b^\mu = -\lambda \nabla^\mu \xi + t^\mu. \tag{39}$$

energy-momentum tensor

$$\begin{aligned}
d_s^\mu &= -\kappa(Du^\mu - \beta\nabla^\mu T) + P_t t^\mu, &(40)\\
d_a^\mu &= -\lambda_a \beta^{-1}(\beta D u^\mu - \beta^2 \nabla^\mu T - 2k^\mu), &(41)\\
e &= \bar{P} - \zeta\theta - (1/3)P_{k\omega}(k^2 + \omega^2), &(42)\\
e_s^{\langle\mu\nu\rangle} &= 2\eta\sigma^{\mu\nu} + P_{k\omega}(k^{\langle\mu}k^{\nu\rangle} + \omega^{\langle\mu}\omega^{\nu\rangle}), &(43)\\
e_a^{[\mu\nu]} &= \gamma\left(\beta\nabla^{[\mu}u^{\nu]} + t^{\mu\nu}\right). &(44)
\end{aligned}$$

spin tensor

$$\begin{aligned}
i^{\lambda\mu} &= -\chi_1 \Delta^{\lambda\mu} u^\beta \nabla^\alpha \omega_{\alpha\beta} - \chi_2 u_\nu \nabla^{\langle\lambda}\omega^{\mu\rangle\nu} &(45)\\
&\quad -\chi_3 u_\nu \Delta^{[\mu}_\rho \nabla^{\lambda]}\omega^{\rho\nu} + \frac{A_3}{2} t^{\lambda\mu}, &\\
j^{\lambda\mu\nu} &= \frac{\chi_4}{2}\nabla^\lambda \omega^{\langle\mu\rangle\langle\nu\rangle} + \frac{A_3}{2}(\Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu), &(46)
\end{aligned}$$

TWO SCALES FOR EXPANSION: GRADIENTS (KNUDSEN NUMBER) & MAGNITUDE OF THE SPIN POLARIZATION TENSOR

4. Summary and Outlook

- a unified (hybrid) approach to spin hydrodynamics is proposed that combines the results of kinetic theory with the IS method
- consistency beetween LKT and IS approaches can be achieved
- technical difficulties of NLKT can be circumvented
- a different starting point compared to THV approach
- second order theory (in gradients) needed

Back-up slides

$N^\mu = n u^\mu$ **current = density $\times$ flow vector**
analogies for energy, momentum and spin

1) conservation of energy and momentum with an asymmetric energy-momentum tensor

$$\boxed{T^{\mu\nu}(x) = g^\mu(x) u^\nu(x)}, \quad \partial_\nu T^{\mu\nu}(x) = 0 \tag{47}$$

u^μ is the four-velocity of the fluid element, while g^μ is the density of four-momentum
with the notation $\partial_\nu (f u^\nu) \equiv Df$ we may write $Dg^\mu = 0$

2) conservation of total angular momentum $J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$ (orbital and spin parts)

$$\boxed{L^{\lambda,\mu\nu}(x) = x^\mu T^{\nu\lambda}(x) - x^\nu T^{\mu\lambda}(x), \quad S^{\lambda,\mu\nu}(x) = s^{\mu\nu}(x) u^\lambda(x)} \tag{48}$$

$s^{\mu\nu} = -s^{\nu\mu}$ describes the spin density

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \rightarrow Ds^{\mu\nu} = g^\mu u^\nu - g^\nu u^\mu \tag{49}$$

3) 10 equations for 13 unknown functions: g^μ, $s^{\mu\nu}$ and u^i $(i = 1, 2, 3)$
additional constraint has been adopted, the Frenkel (or Weyssenhoff) condition
$s^{\mu\nu} u_\mu = 0$

ideas still frequently cited in the context of the Einstein-Cartan theory

Pseudo-gauge transformation (QCD language in the context of the proton spin puzzle: adding boundary terms)

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda}\right) \tag{50}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho} \tag{51}$$

One most often considers free Dirac field, should describe a gas of fermions, good starting point for thermodynamics and/or hydrodynamics

Canonical forms (directly obtained from Noether's Theorem): asymmetric energy-momentum tensor, spin tensor directly expressed by axial current (couples to weak interactions)

Belinfante-Rosenberg version, $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}, Z^{\mu\nu,\lambda\rho} = 0$, (couples to classical gravity); spin tensor appears in modified theories of gravity, couples to torsion

de Groot, van Leuveen, van Weert (GLW) forms: symmetric energy-momentum tensor and conserved spin tensor

Hilgevoord and Wouthuysen (HW) choice: symmetric energy-momentum tensor and conserved spin tensor