UNIVERSITÄT BIELEFELD

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Lee-Yang edge singularities from analytic continuation of scaling functions

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work done with F. Karsch *and* C. Schmidt

based on Phys.Rev.D 109 (2024) 1, 014508

Aspects of Criticality II -

EMMI Workshop @ University of Wrocław

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Outline

I. Motivation - QCD phase diagram

II. Anatomy of Yang-Lee zeros and edge singularities

III. Scaling functions from lattice simulations

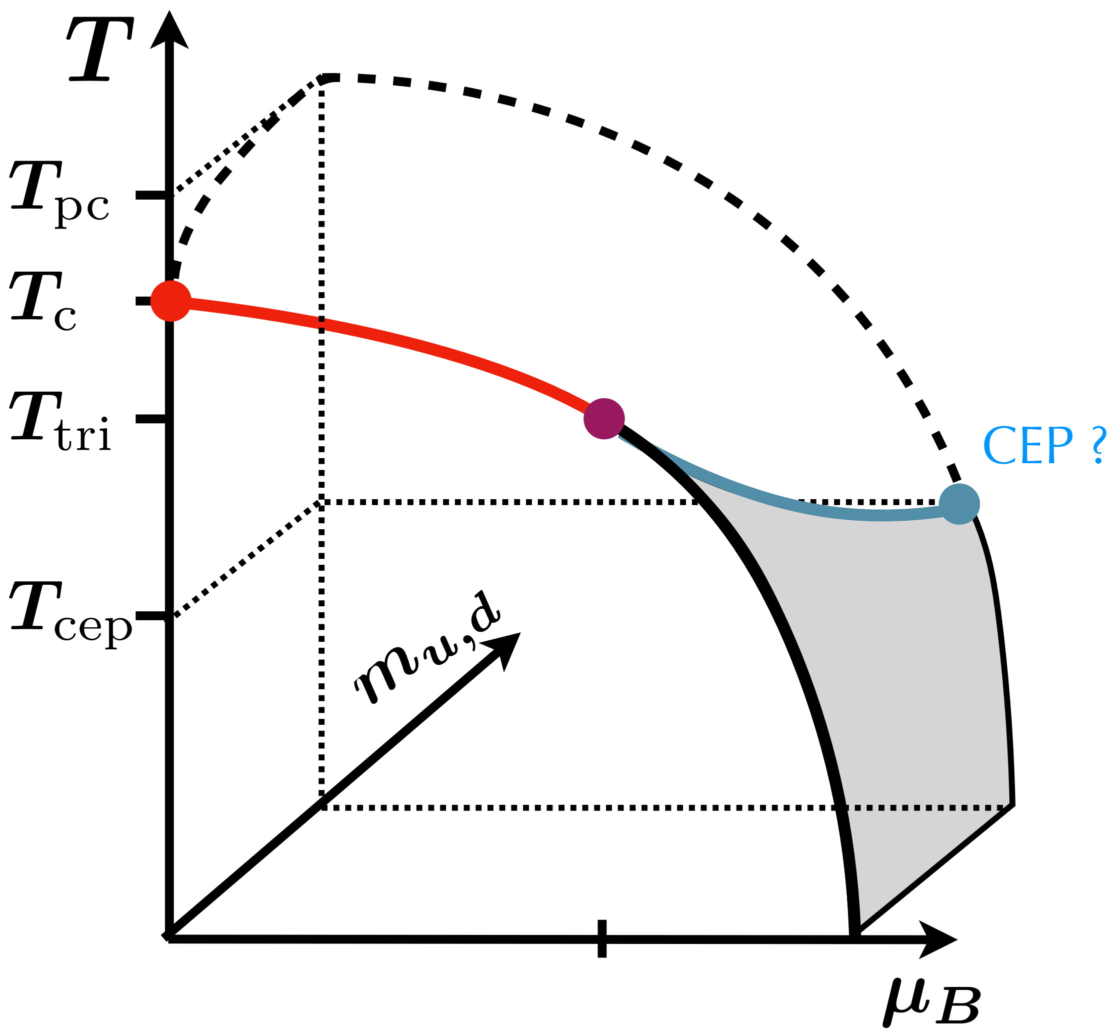
IV. Schofield parametrisation and Lee-Yang edges

V. Results for universal location of YLEs

VI. Relevance for the QCD phase diagram

VII. Summary

Motivation - QCD phase diagram



Conjectured phase diagram
[F. Karsch, arXiv:2212.03015]

CEP - an open problem because :

- Experimentally hard - system dynamic and short lived
- Strongly interacting at low energy
- Lattice has sign problem at finite μ_B
- In FRG one has truncations …

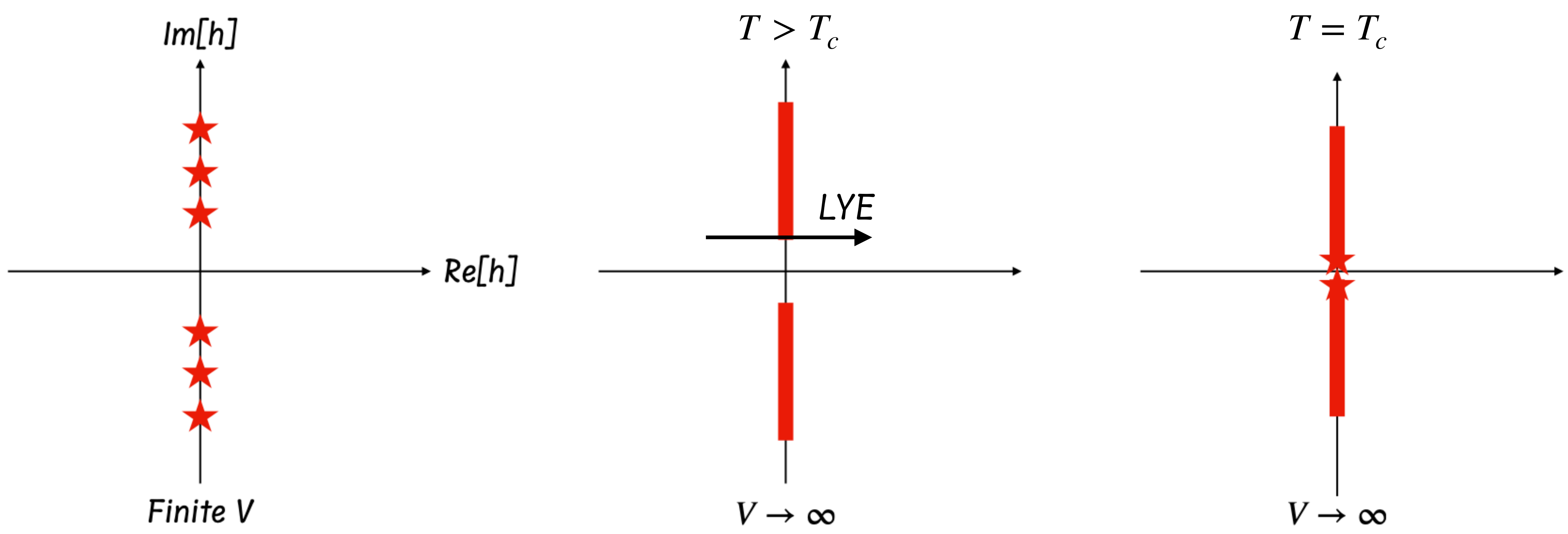
Yet, a lot of progress in studying the phase diagram comes from lattice studies of QCD

- Taylor expansions about $\mu_B = 0$, [Allton, et.al Bielefeld-Swansea (2002)]
- Analytic continuation from imaginary μ_B simulations [de Forcrand, Philipsen (2002)], [D'Elia and Lombardo, (2002)]

Yang-Lee zeros and edge singularities

- Taylor expansions limited by complex singularities of the partition function!

  [C. N. Yang, T. D. Lee, Phys. Rev., vol. 87, 1952]



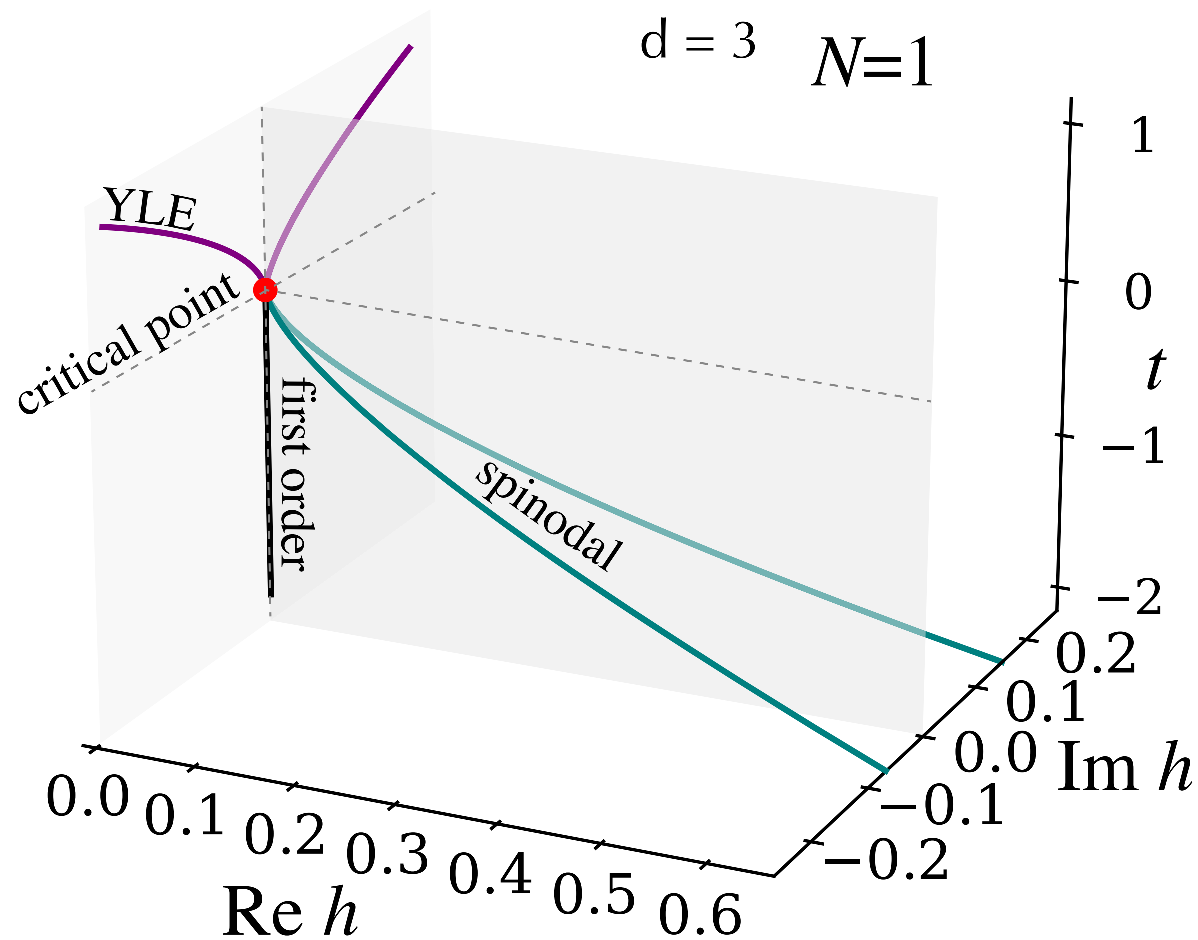
LY zeros for the Ising model

- Gap in density of zeros in symmetric phase - closest zero is the Yang-Lee edge - described by ϕ^3 theory with imaginary coupling

  [M. E. Fisher, Phys. Rev. Lett. 40 (1978)]

Yang-Lee zeros and edge singularities

- LYE is a critical point with one relevant scaling direction $M - M_c \sim (h - h_c)^\sigma$
- Moreover, continuously connected to the standard critical point at $t = 0$, $h = 0$
- Images of the Spinodal points in the $T < T_c$ branch of the EoS [X. An, D. Mesterházy, and M. A. Stephanov, J. Stat. Mech., (2018)]

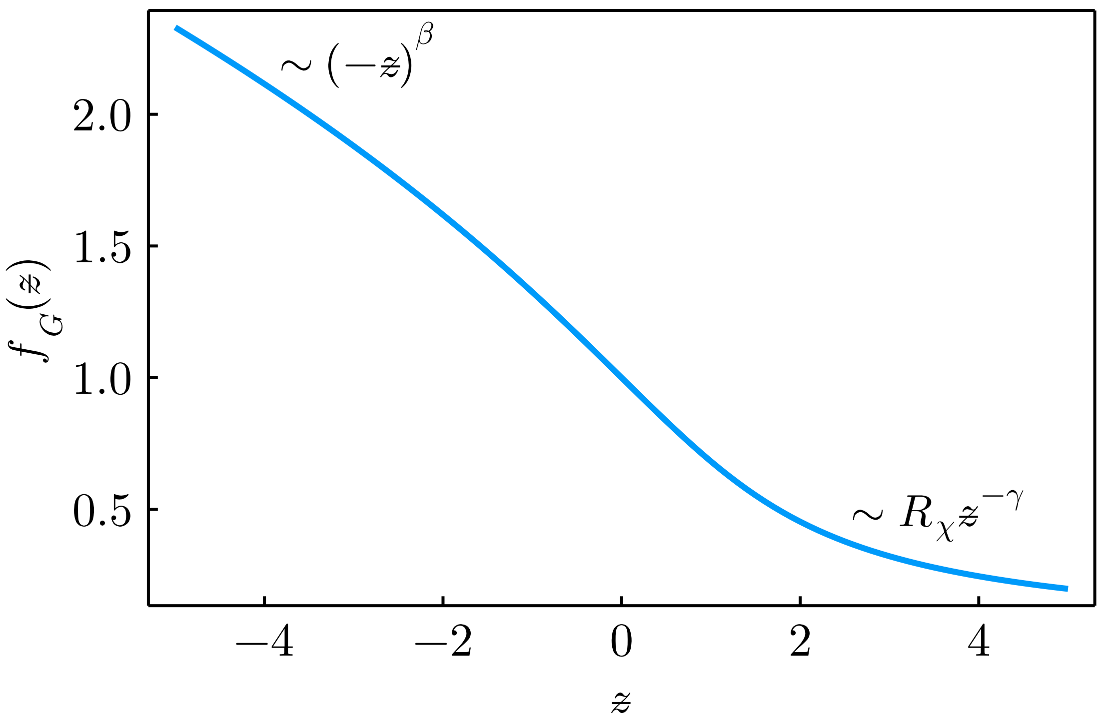


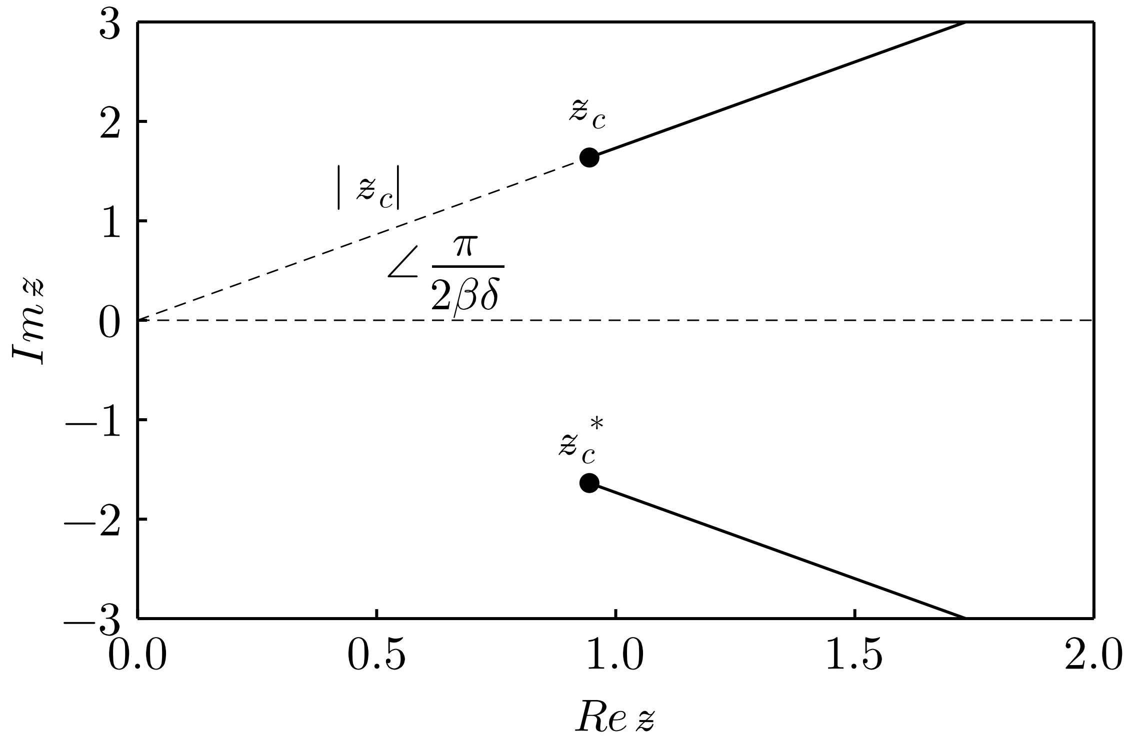
[G.Johnson, F. Rennecke & V. V. Skokov Phys.Rev.D 107 (2023)]

Yang-Lee zeros and edge singularities

- Location of LYE $z_c \equiv \frac{t}{h_{LYE}^{1/\beta\delta}(t)}$, is universal, $f_G(z) = (z - z_c)^\sigma$

- Phase is determined from *Circle Theorem* in Ising model but $|z_c|$ not





[F. Rennecke and V. V. Skokov Annals Phys. 444 (2022)]

- ϵ expansion around $d = 4$ not an option because LYE described by ϕ^3 theory

- Until recently only results from FRG were present.

Scaling functions from lattice simulations

- Scaling functions in $3-d$ $O(N)$ for $N \in \{1,2,4\}$ and their finite size dependence studied using Monte Carlo simulations

- New MC simulations with improved ϕ^4 models aimed at suppressing corrections to scaling performed for $3-d$ $O(2)$ and $Z(2)$ universality classes [F. Karsch, M. Neumann, M. Sarkar in Phys. Rev. D 108, (2023)]

- For $O(4)$, still an un-improved Hamiltonian but updated parametrization from [J. Engels and F. Karsch Phys. Rev. D 85 (2012)]

- Quantity of interest in the following : $M = -\dfrac{\partial f}{\partial H} \equiv h^{1/\delta} f_G(z)$

- The LYEs would then be located at zeros of the inverse magnetic field susceptibility $\dfrac{1}{f'_G(z_c)} = \dfrac{dz(f^c_G)}{df^c_G} = 0$

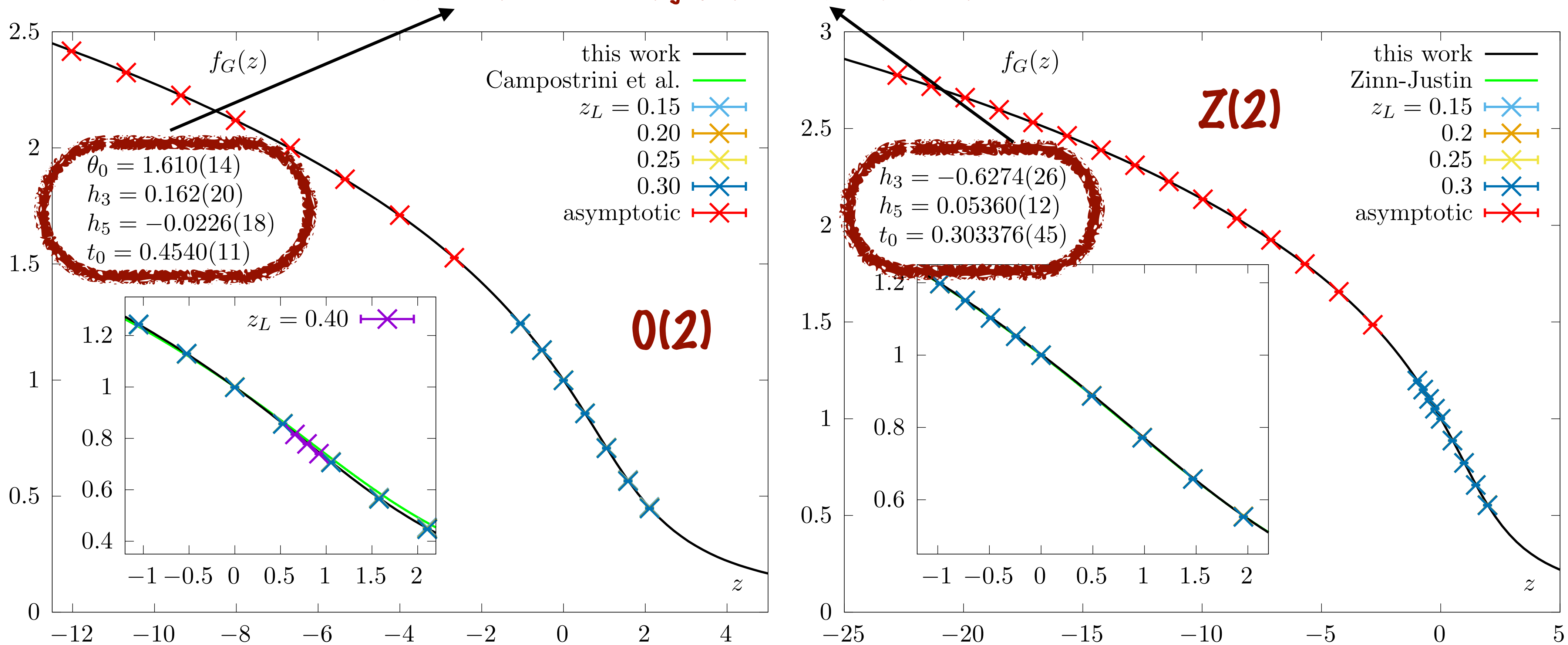
Scaling functions from lattice simulations

- Using the Schofield [P. Schofield, Phys. Rev. Lett. 22, 1969] parametrization of the Widom-Griffiths form,

$$M = m_0 R^{\beta}\theta \; , \; t = R(1-\theta^2) \; , \; h = h_0 R^{\beta\delta} h(\theta)$$

the following fit parameters were obtained*

! Calculated on Lattice using Monte Carlo simulations !



*[F. Karsch, M. Neumann, M. Sarkar in Phys. Rev. D 108, (2023)]

Schofield parametrisation and Lee-Yang edges

- Under this parametrization, $(t, h) \to (R, \theta)$, with R defining the distance from the critical point.

- Thermodynamic functions are analytic functions of θ

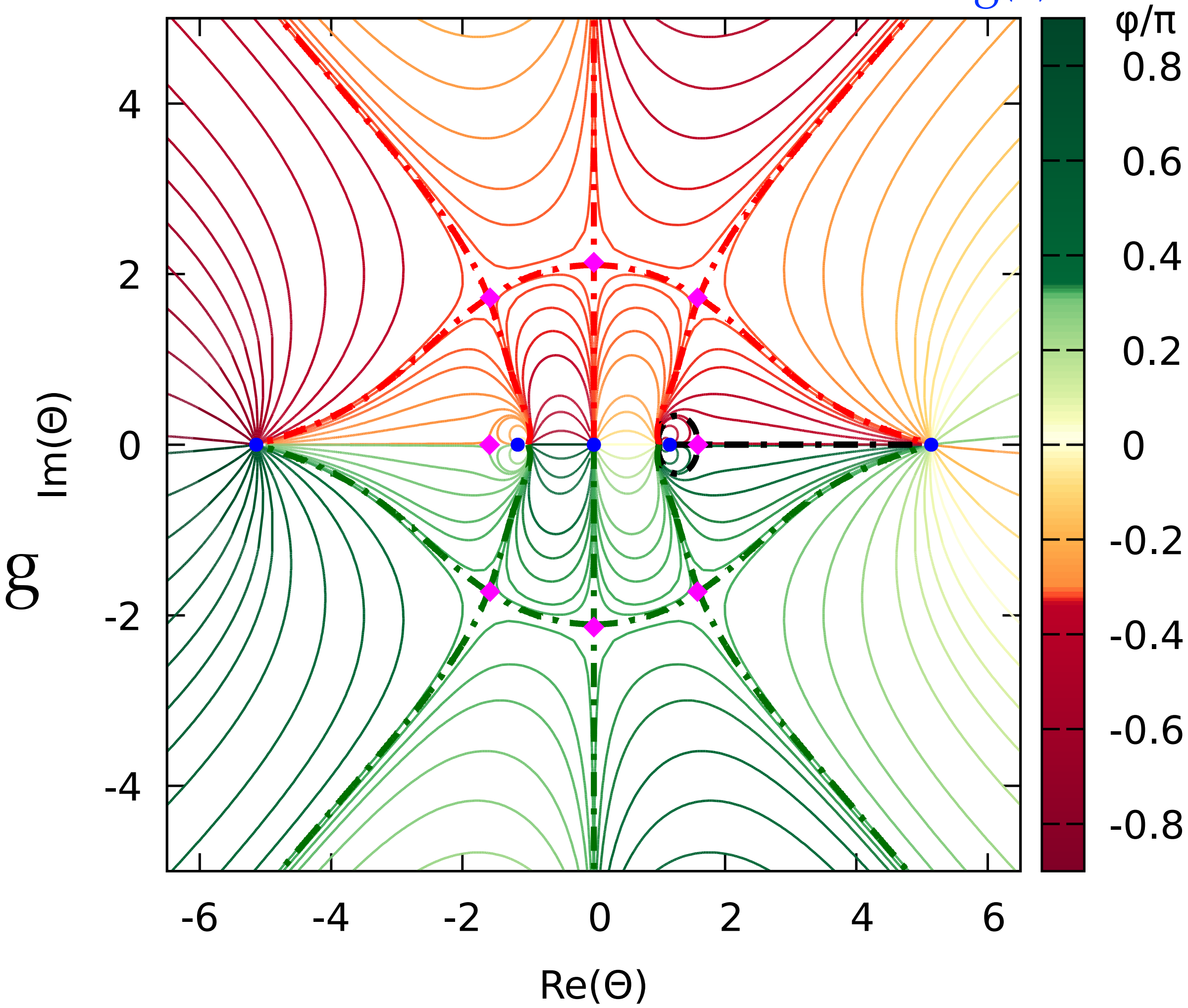
- The scaling function takes the form

$$f_G(\theta(z)) = \theta \left(\frac{h(\theta)}{h(1)} \right)^{-1/\delta},$$

- with $h(\theta)$ odd polynomial and the scaling variable $z = t/h^{1/\beta\delta}$ given by,

$$z(\theta) = \frac{1-\theta^2}{\theta_0^2 - 1} \theta_0^{1/\beta} \left(\frac{h(\theta)}{h(1)} \right)^{-1/\beta\delta}$$

$3d - Z2$: Lines of constant arg(z)



[F. Karsch, S.S., C. Schmidt *Phys.Rev.D* 109 (2024)]

Schofield parametrisation and Lee-Yang edges

- Immediate consequence of the map : points on Im[θ] axis have the LY phase

- Entire real z axis mapped between $\theta \in (0,\theta_0)$ with $z = 0$ at $\theta = 1$, $z \to +\infty$ at $\theta = 0$ and $z \to -\infty$ at $\theta = \theta_0$

- The location of the LYE is obtained by solving

$$\frac{dz(\theta)}{df_G(z)} = 0 \implies \frac{dz(\theta)}{d\theta} = 0 \qquad 0 = 2\beta\delta\theta h(\theta) - \left(\theta^2 - 1\right) h'(\theta)$$

- We find additional zeros than the expected LYEs

- The map $z(\theta) \mapsto \theta$ is not invertible and we need to find the region where the map is well defined.

- Verify that $f_G(z_c)$ has a branch cut originating from z_c and its complex conjugate

Analysis for $3-d\ Z2$

- Using the critical exponents : $\beta = 0.32643(7)$ and $\delta = 4.78982(85)$ from [M. Hasenbusch, Phys. Rev. B 100,(2019)]

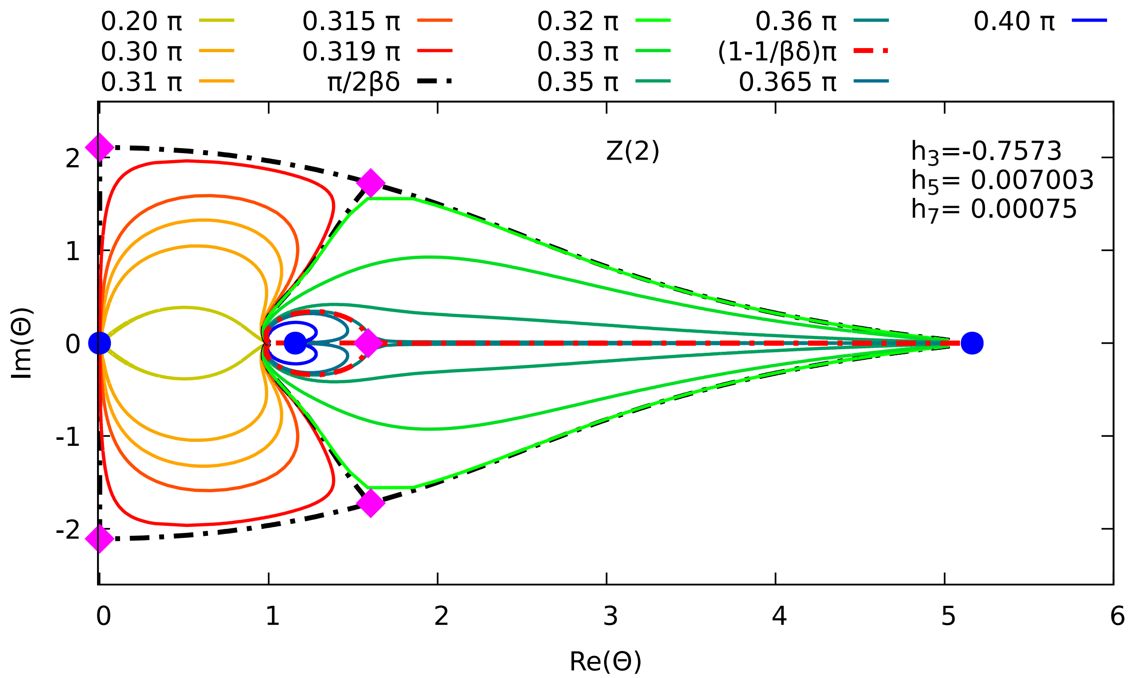
- Ansatz for $h(\theta) = \left(\theta + h_3\,\theta^3 + h_5\,\theta^5 + h_7\,\theta^7\right)$

- Location of the LYEs : $|z_c|e^{i\phi}$, with $|z_c| = 2.418(55)$ and $\phi = 0.9935^{+45}_{-191}\phi_{LY}$

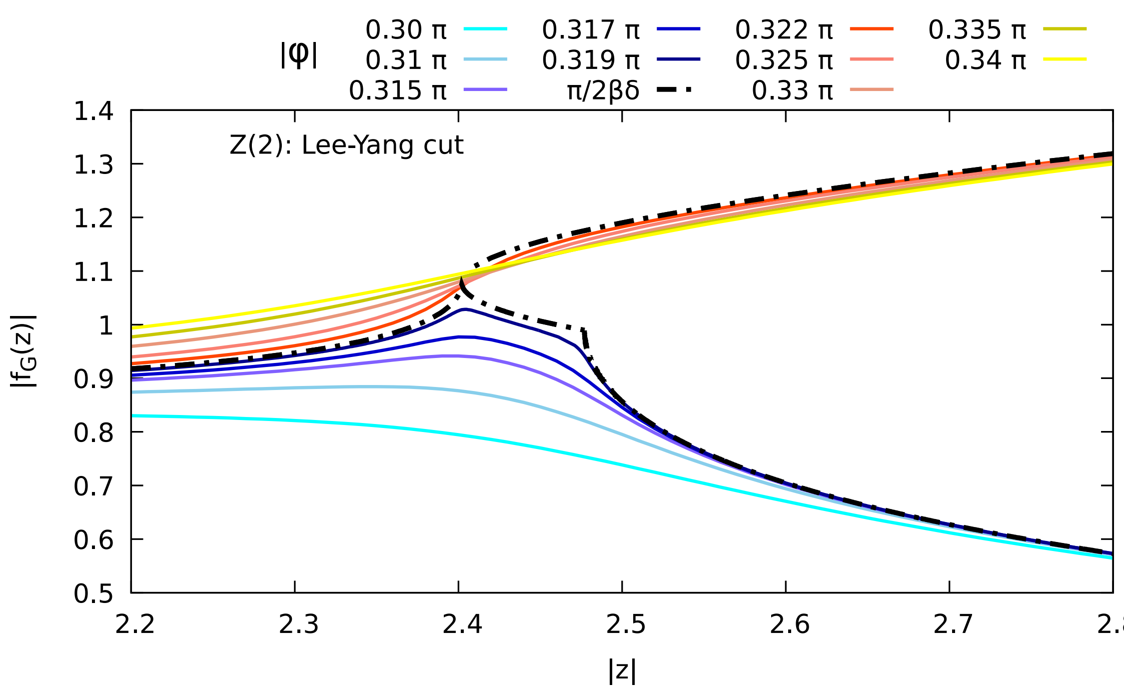
  where $\phi_{LY} = \dfrac{\pi}{2\beta\delta}$

- Branch cut in $f_G(z)$ along the branch cut

$3d - Z2$: Lines of constant $\arg(z)$, region where map is single valued



$3d - Z2$: Branch cut in $f_G(z)$



[F. Karsch, S.S., C. Schmidt *Phys.Rev.D* 109 (2024)]

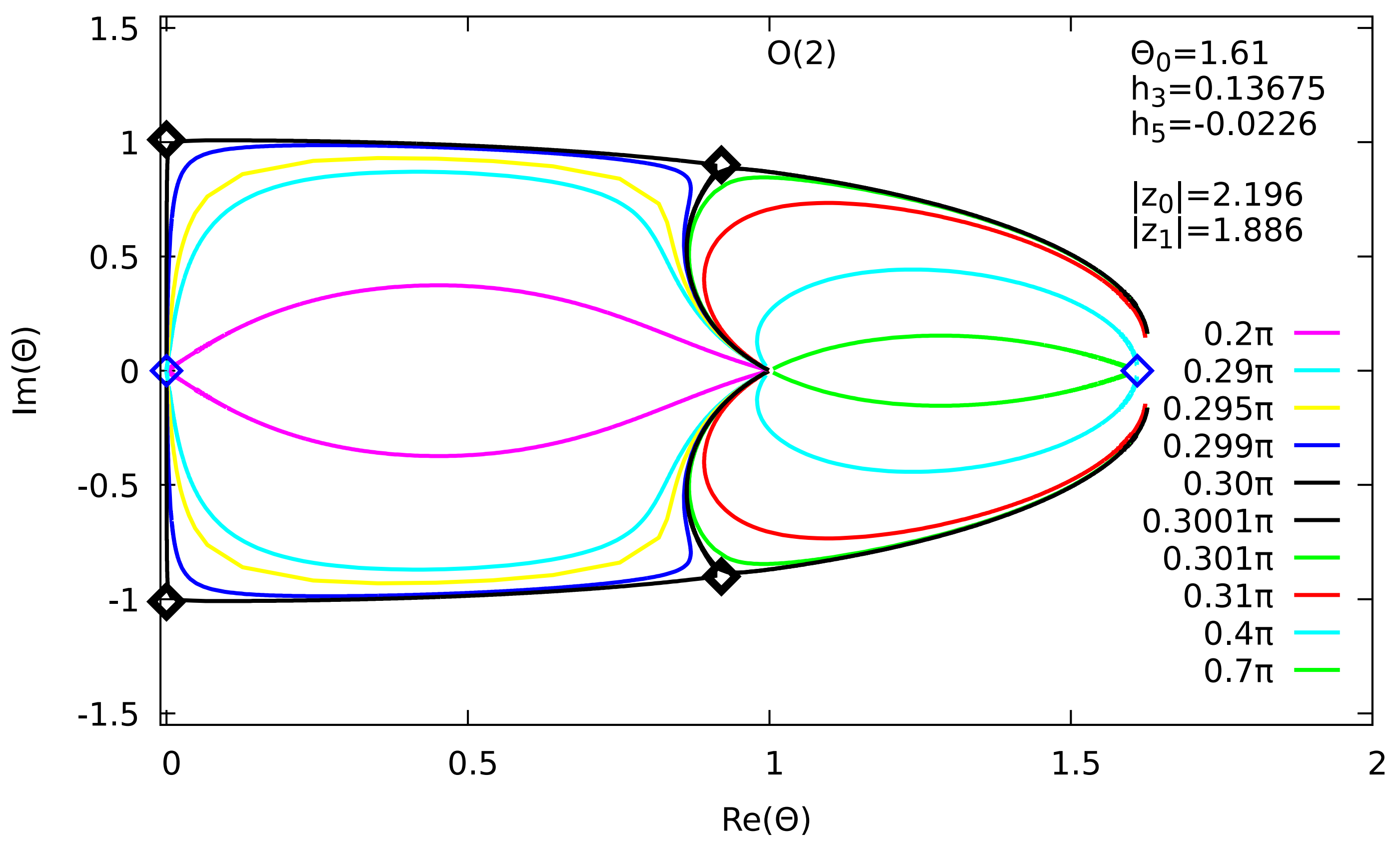
Analysis for $3-d\ O(2)$

- Using the critical exponents : $\beta = 0.34864(7)$ and $\delta = 4.7798(5)$ from [M. Hasenbusch, Phys. Rev. B 100,(2019)]

- Ansatz for $h(\theta) = \left(\theta + h_3\,\theta^3 + h_5\,\theta^5\right) \times \left(1 - \theta^2/\theta_0^2\right)^2$ now includes the Goldstone modes

- Location of the LYEs : $|z_c| = 1.900(46)$, $\phi/\phi_{LY} = 1.024(30)$

$3d - O(2)$: Lines of constant $\arg(z)$, region where map is single valued



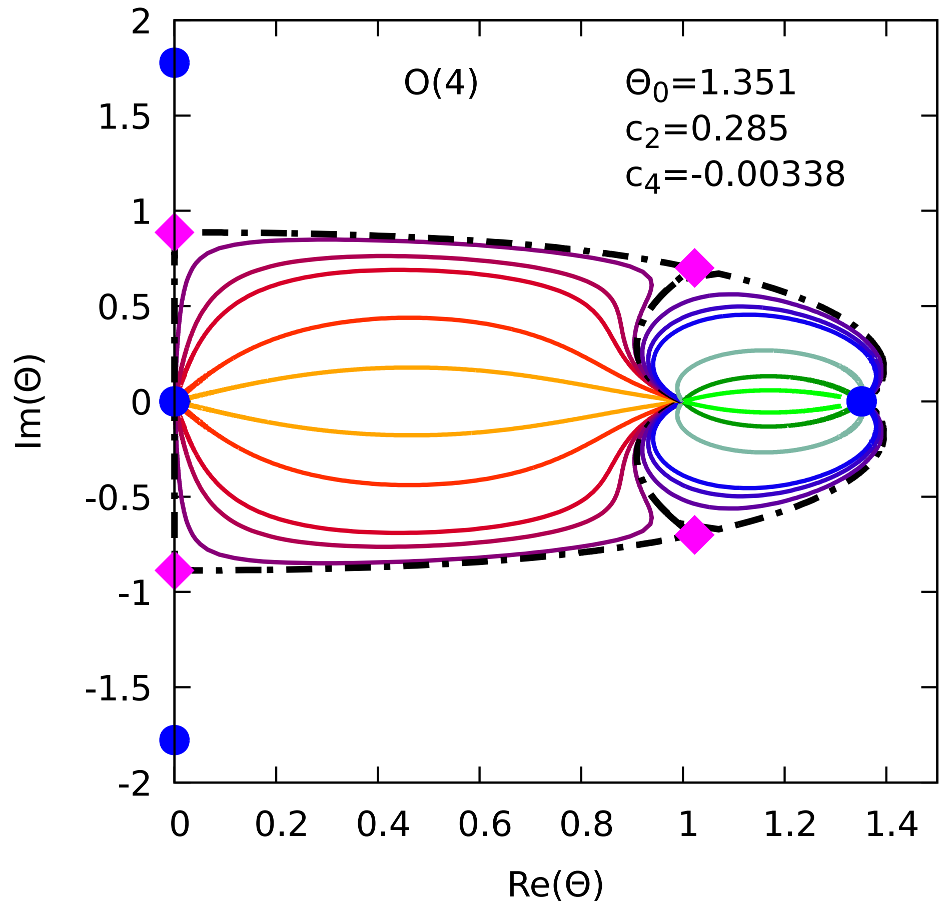
Analysis for $3-d\ O(4)$

- Using the critical exponents : $\beta = 0.380(2)$ and $\delta = 4.824(9)$ from [J. Engels, L. Fromme, and M. Seniuch, Nucl. Phys. B 675, (2003)]

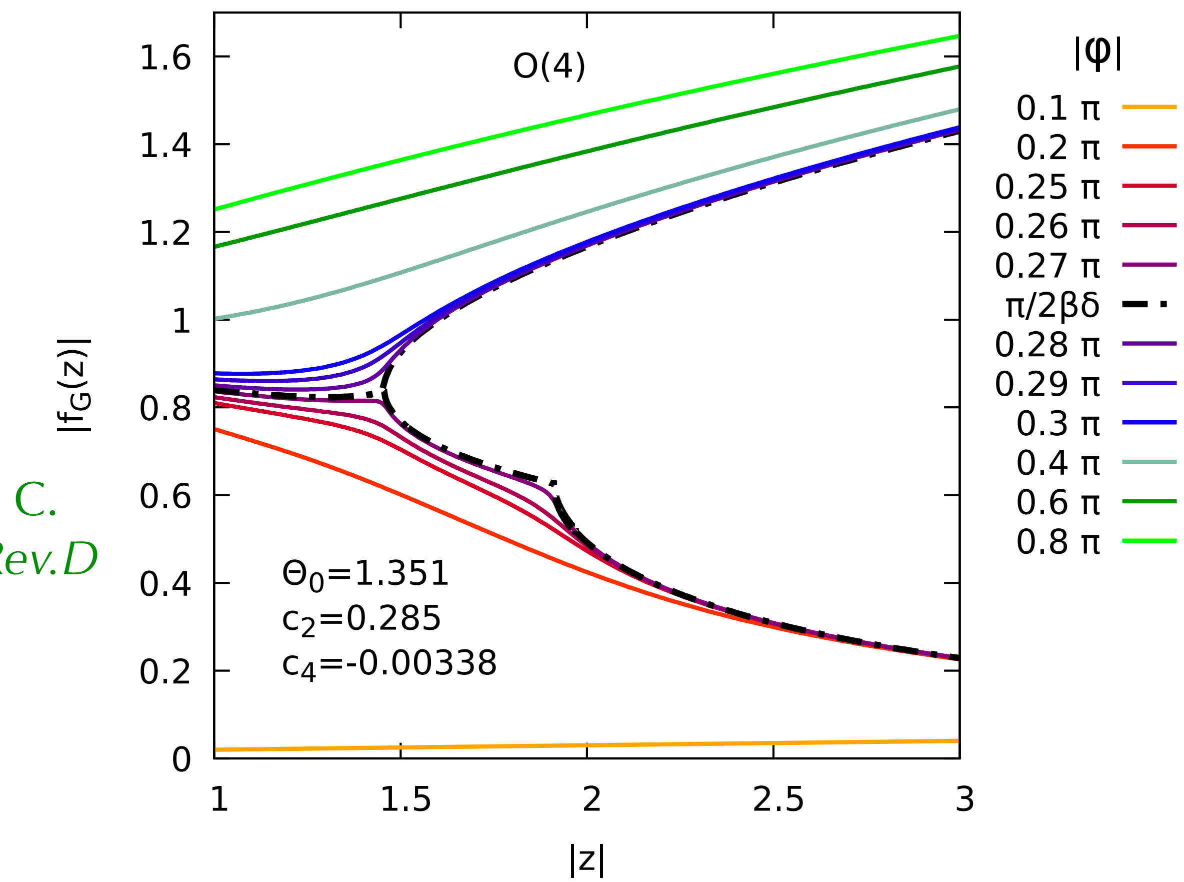
- Ansatz for $h(\theta) = \left(\theta + h_3\,\theta^3 + h_5\,\theta^5\right) \times \left(1 - \theta^2/\theta_0^2\right)^2$, same as $O(2)$ with different parameters

- Locations of LYEs : $|z_c| = 1.469(20)$, $\phi/\phi_{LY} = 1.023(34)$

$3d - O(4)$: Branch cut in $f_G(z)$



[F. Karsch, S.S., C. Schmidt *Phys.Rev.D* 109 (2024)]



$3d - O(2)$: Lines of constant $\arg(z)$, region where map is single valued

Analysis for mean field approximation and $N \to \infty$

- In the MFA ($g = 1$) and $N \to \infty$ ($g = 2$) the scaling function and critical exponents are given by

$$f_G(z + f_G^2)^g = 1 \qquad \beta = \frac{1}{2}\ , \quad \delta = \begin{cases} 3\ , \mathrm{MFA} \\ 5\ , N \to \infty\ 3 - d \end{cases}$$

- We can determine LYEs using the Schofield parametrization

$$\tilde{\theta}_\pm = \pm \frac{\theta_0}{\sqrt{\delta - (\delta - 1)\theta_0^2}}$$

- Substituting $\tilde{\theta}_\pm$ into the equation for $z(\theta)$, the universal location of the Lee-Yang edge singularity in the complex $z-$plane is obtained as

$$z_{LY} \equiv z(\tilde{\theta}_\pm) = \begin{cases} 3 \cdot 2^{-2/3}\ \mathrm{e}^{\pm i\pi/3}\ ,\ \ \mathrm{MFA} \\ 5 \cdot 2^{-8/5}\ \mathrm{e}^{\pm i\pi/5}\ ,\ \ N \to \infty\ 3 - d \end{cases}$$

Comparison of results with FRG

- The only other results that exist for $|z_c|$ are from FRG [G. Johnson et. al. Phys. Rev. D 107, (2023)]

	Z(2)	O(2)	O(4)
Lattice	2.429(56)	1.95(7)	1.47(3)
FRG	2.43(4)	2.04(8)	1.69(3)

Relevance for Lattice QCD

- Gives us another **universal quantity more suited for finite volume** studies.

- Being the closest complex singularity in parameter space - **will effect the Taylor series coefficients**! [G.Basar, G.V. Dunne Z. Yin, *Phys.Rev.D* 105 (2022)]

Summary

- Using the Schofield parametrization, it is possible to **extract the universal location of Lee-Yang edge singularities** from lattice based studies - **first results** from lattice based studies.

- We considered the scaling functions in $3-d\ \ O(N=1,2,4,\infty)$ universality class and MFA.

- A discussion on the region in the complex θ plane, where the function $z(\theta)$ is invertible was also discussed for all the cases.

- Various choices for the order of the polynomial $h(\theta)$ studied - **results stable and consistent with FRG** (except O(4))

Outlook

- The choice of a **truncated polynomial ansatz for** $h(\theta)$ **needs to be refined** - in order to study how these edge singularities are approached.

BACKUP SLIDES

Higher Taylor coefficients and LYE

[G.Basar, G.V. Dunne Z. Yin, *Phys.Rev.D* 105 (2022)]

$d = 3 \quad \sigma_{MF} = \frac{1}{2} \quad \sigma = 0.085(1)$

conformal bootstrap

In general since $\mu^2 = \mu^2_{\rm LY}$ is the closest singularity to the origin, the radius of convergence of the Taylor expansion in Eq. (15) is $|\mu^2_{\rm LY}|$. However the coefficients $c_n(T)$ contain much more information than just the radius of convergence. The Darboux theorem [30,31] states that the behavior of the coefficients $c_n(T)$ at large order n is directly related to the behavior of the function in the vicinity of the nearest singularity. Specifically, if the Taylor expansion coefficients of a function $f(z) = \sum_{n=0}^{\infty} b_n z^n$ near the origin have leading large-order growth as $n \to \infty$:

$$b_n \sim \frac{1}{z_0^n}\left[\binom{n+g-1}{n}\phi(z_0) - \binom{n+g-2}{n} z_0 \phi'(z_0) + \binom{n+g-3}{n}\frac{z_0^2}{2!}\phi''(z_0) - \cdots\right] \tag{16}$$

then the leading singularity is located at z_0, and in the vicinity of z_0 the function behaves as

$$f(z) \sim \phi(z)\left(1 - \frac{z}{z_0}\right)^{-g} + \psi(z), \qquad z \to z_0 \tag{17}$$

where $\phi(z)$ and $\psi(z)$ are analytic near z_0. This means that from a detailed study of the expansion coefficients $c_n(T)$, derived from the expansion about $\mu = 0$, we can learn about the expansion of the function near the critical point.

Langer cut in $3d - Z2$

$H < 0, T < T_c$

[X. An, D. Mesterházy, and M. A. Stephanov, J. Stat. Mech., (2018)]

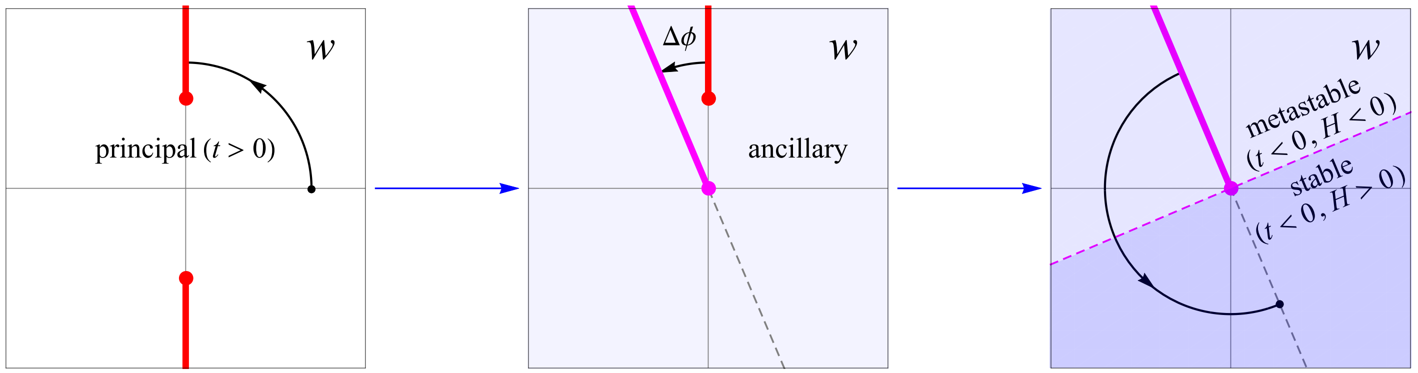
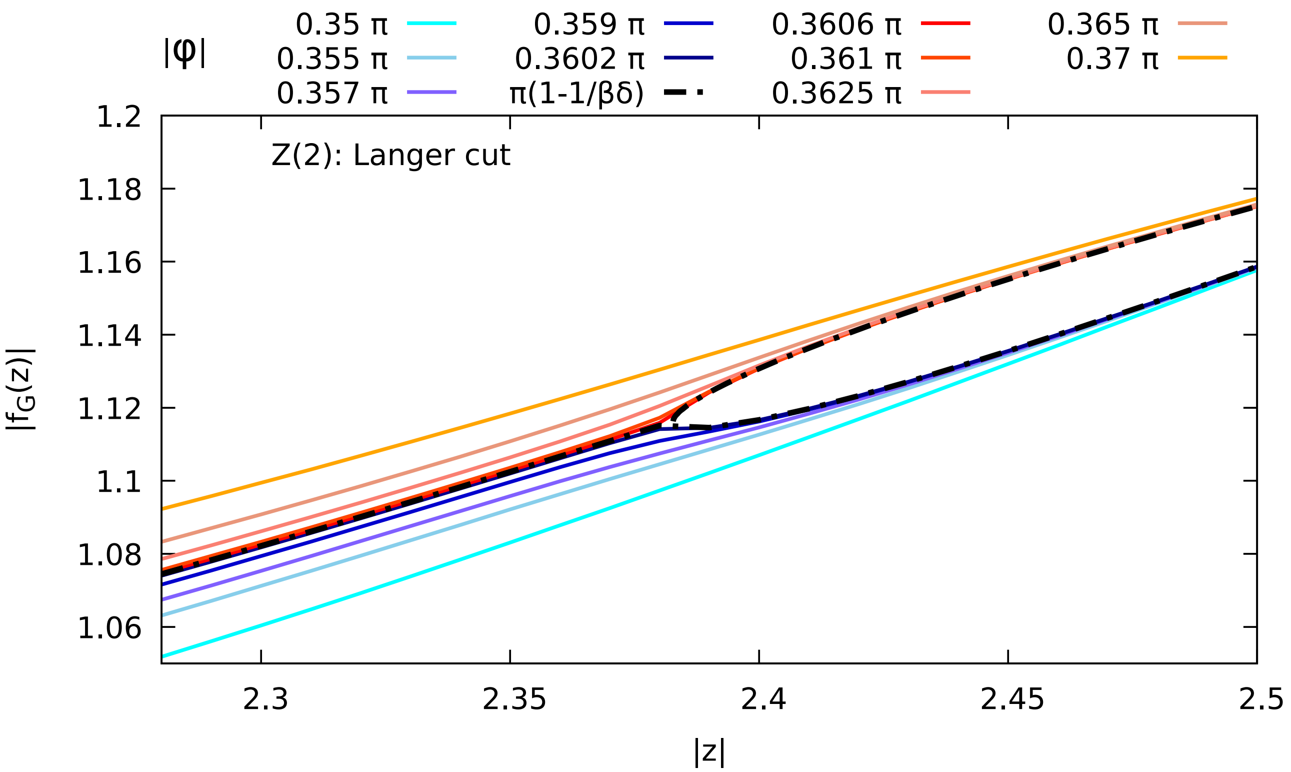


Figure 3: Analytic continuation $t \to -t$ from the principal, i.e., high-temperature sheet (left panel) to the low-temperature sheet (right panel) of the scaling function $z(w)$ of the Ising theory as conjectured by Fonseca and Zamolodchikov, where $w \sim Ht^{-\beta\delta}$, while keeping the magnetic field $H > 0$ fixed at $d = 4 - \varepsilon$. After analytic continuation the metastable branch $H < 0$ can be accessed by rotating H clockwise in the complex plane, while keeping $t < 0$ fixed. The line representing the Langer cut is rotated away from imaginary axis by an angle $\Delta\phi$, cf. Eq. (3.2).

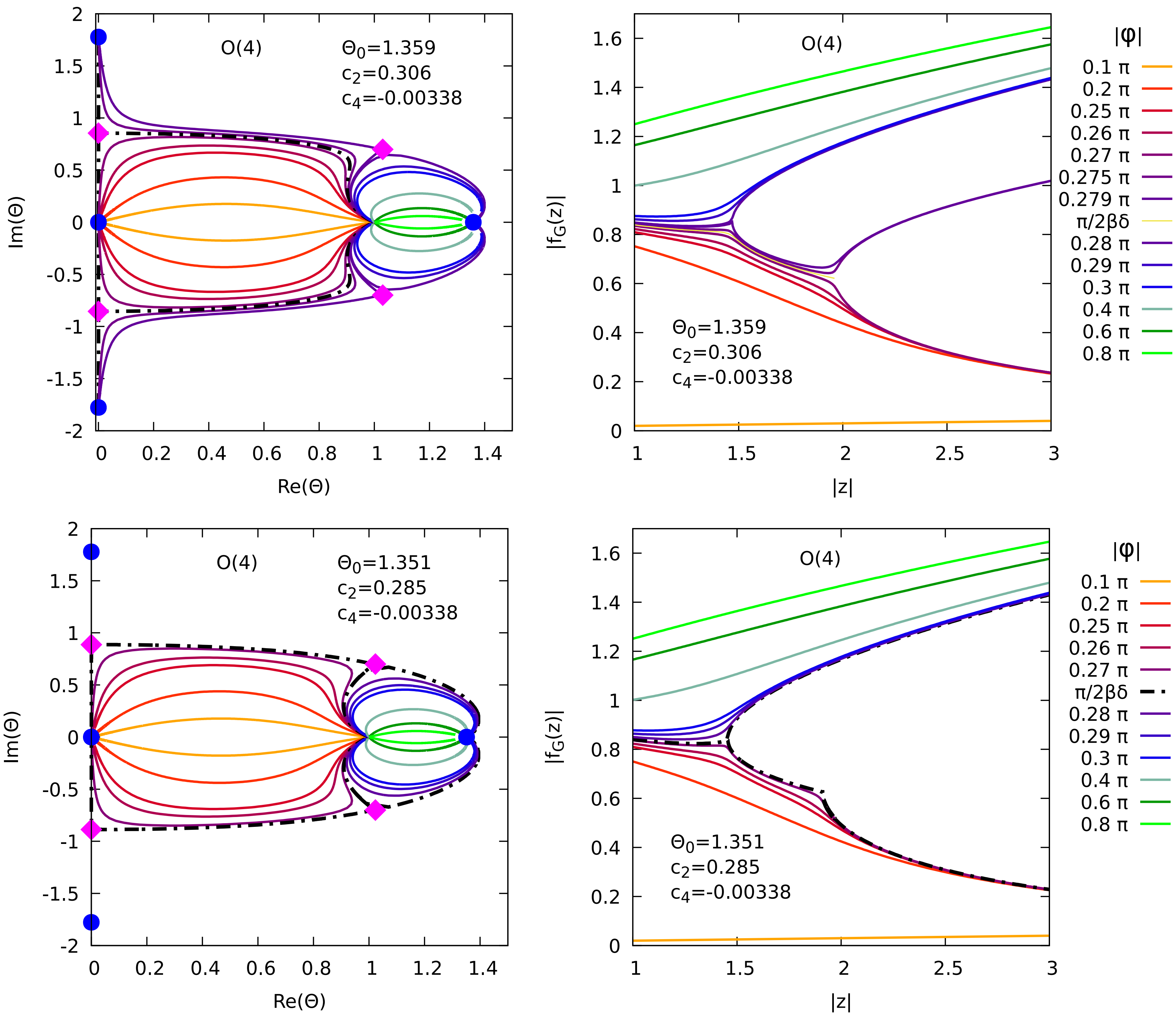


[F. Karsch, S.S., C. Schmidt *Phys.Rev.D* 109 (2024)]

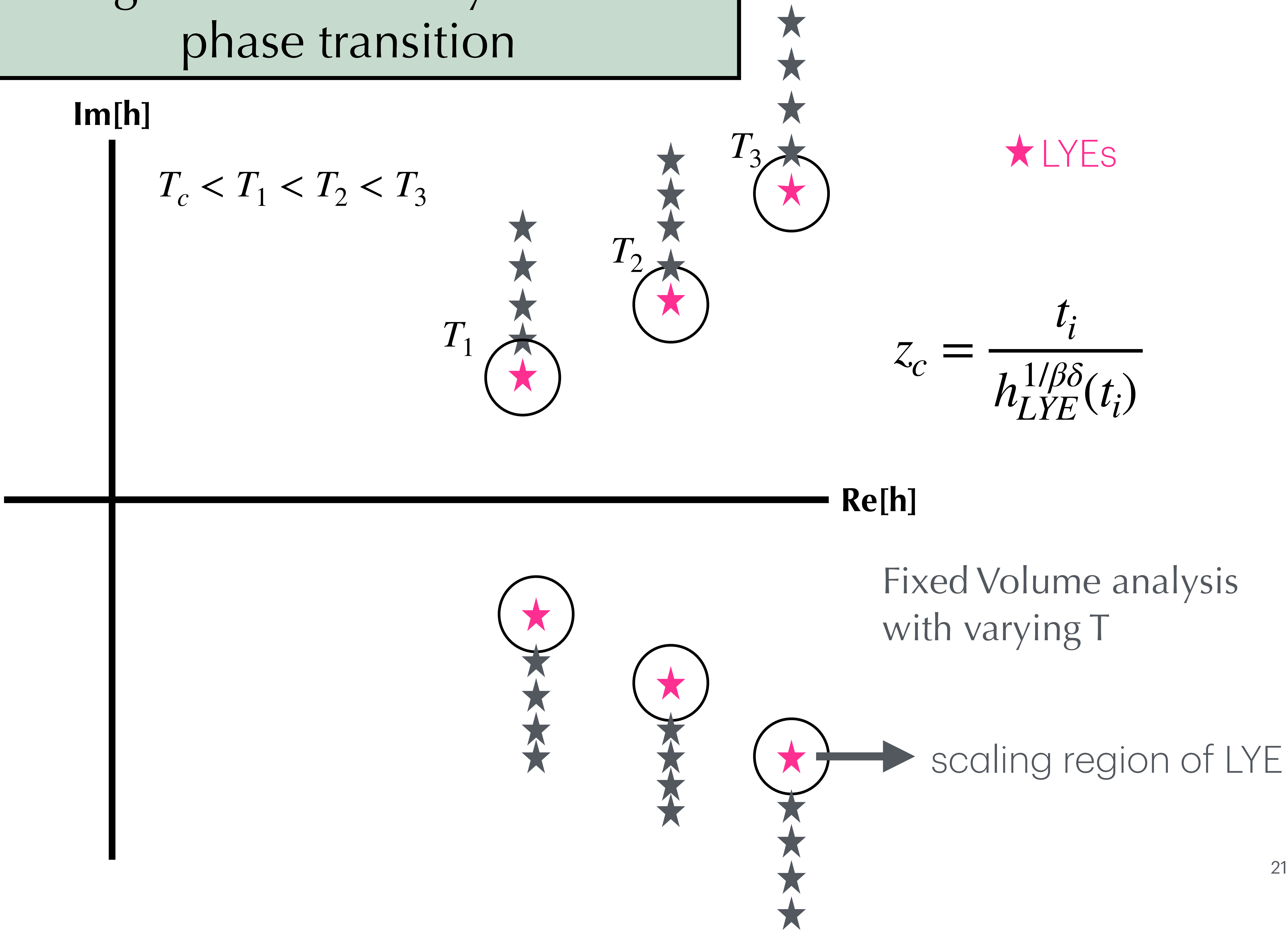
$$z_{LY} = 2.418(55)\ e^{\pm i r \phi_{LY}} \ , \ r = 0.9935^{+45}_{-191} \ ,$$
$$z_{Lan} = 2.379(36)\ e^{i\phi_{Lan}} \ .$$

Tuning of parameters in $3d - O(4)$

[F. Karsch, S.S., C. Schmidt *Phys.Rev.D* 109 (2024)]



Scaling of LYEs when system has a phase transition



$$z_c = \frac{t_i}{h_{LYE}^{1/\beta\delta}(t_i)}$$

Fixed Volume analysis with varying T

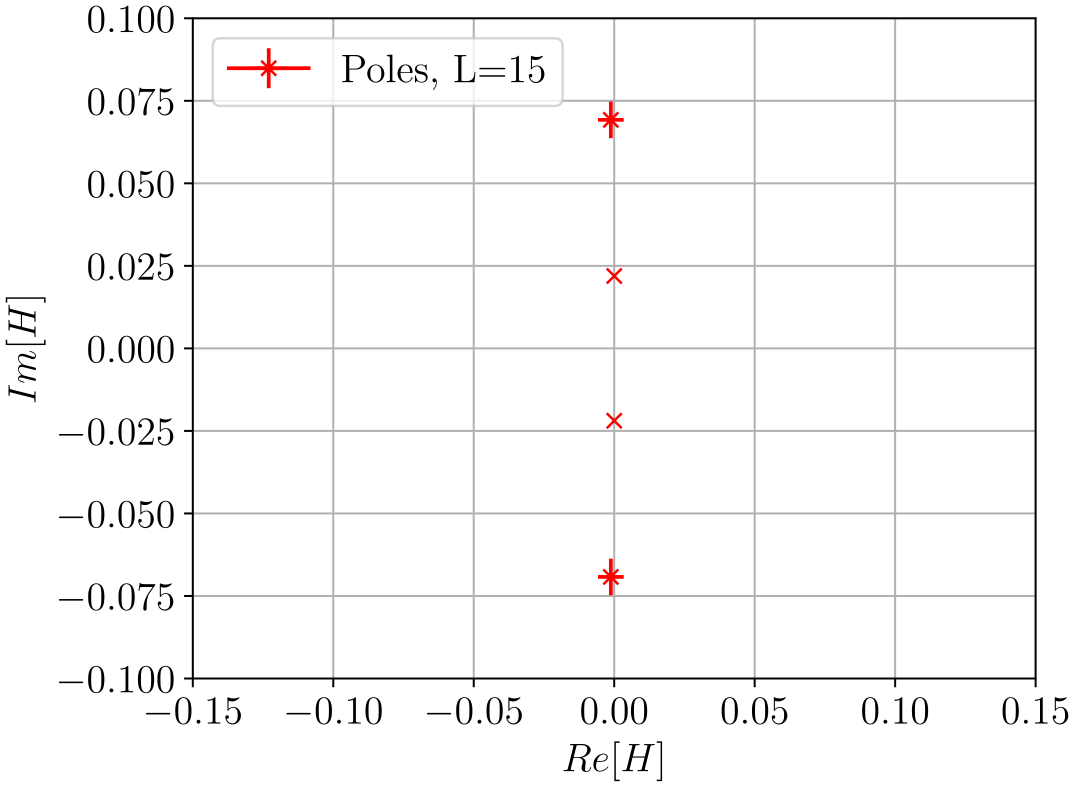
Complex zeros of partition function

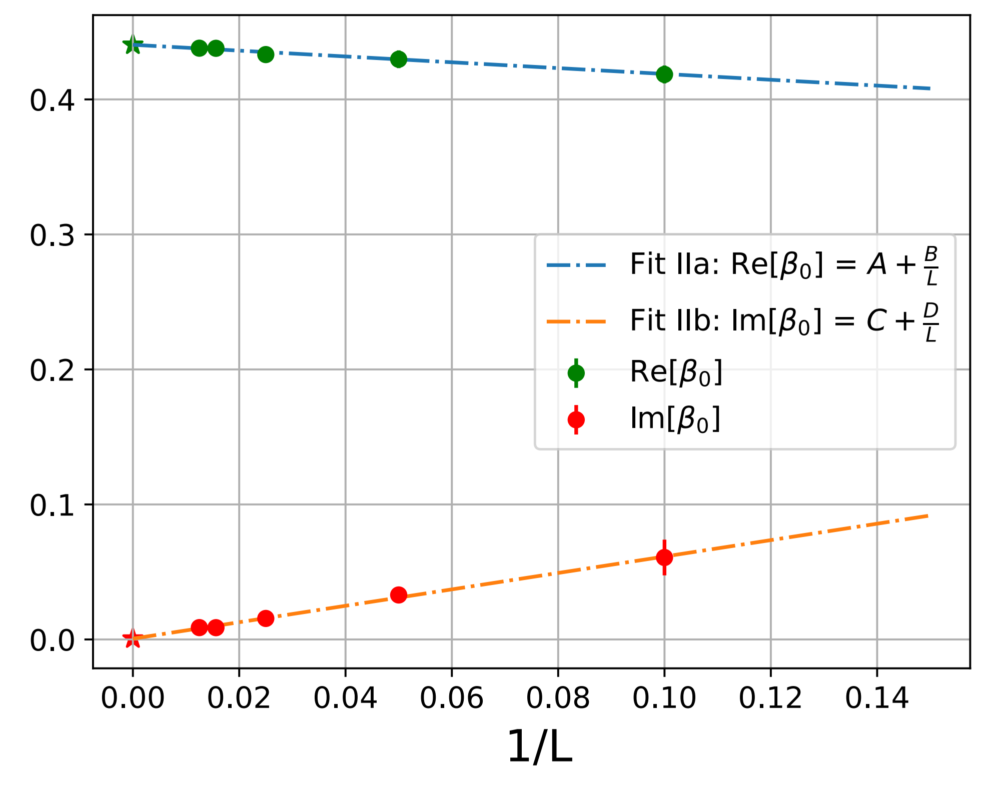
$$\mathcal{Z}_{GC} \equiv Z(V,T,\mu) = \sum_{N=0}^{\infty} c_N(T,V)\, e^{-\frac{\mu}{T}N} \quad , c_N > 0$$

For finite V, real T and μ_B : View as a polynomial of fugacity $z = e^{\mu/T}$ to see $\mathcal{Z}_{GC} > 0$

Case of $2D-$Ising

[S.S., M. Cipressi, F. Di Renzo, *Phys.Rev.D* 109 (2024)]





Quantity	Fit Results	Exact value	χ^2/dof
ν	1.014(60)	1	1.3
β_c	0.4404(19)	~ 0.4407	1.44
$\beta\delta$	1.881(93)	1.875	1.2